ON CERTAIN PROBLEMS OF APPROXIMATION

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

BY

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ALIGARH MUSLIM UNIVERSITY,

ALIGARH

1981

D E D I C A T E D

TO

MY WIFE

MRS AKHTARI KHAN.

PREFACE

The present Thesis entitled 'On certain problems of approximation' embodies the result of my researches which I have been doing since May, 1976 at Aligarh Muslim University, Aligarh. The work has been done under the esteemed guidance and supervision of Dr. Sarfaraz Umar, Professor in Mathematics, Aligarh Muslim University, Aligarh.

The Thesis consists of six chapters. In the first chapter we give a resume of the hitherto known results which have interconnections with our investigations. Chapter II is concerned with the study of the degree of approximation of functions belonging to the class $k(t)$ by Gauss-weierstrass and Generalised Gauss-weierstrass singular integrals, while chapter III deals with the degree of approximation of Picard, Poisson-cauchy and ostrowski's operators for the same class. Chapter IV is devoted to the determination of degree of approximation by Gauss-weierstrass singular integral for the functions belonging to the newly defined classes $K(t)$ and $g(t)$.

In chapter V, we have used more general operater (named Generalised Gauss-weierstrass singular integral) and obtained the degree of approximation generalising the results of chapter IV. Chapter VI contains four theorems with corollaries for Picard and Ostrowski's operator for the functions belonging to the classes h(t) and g(t).

Towards the end, we have given a complete bibliography of research publications which have been referred to this thesis (please see Appendix A).

All the results in this thesis have been communicated for publication in various journals of international reputes, quite a many of which have already been accepted for publication.

I have great pleasure in taking this opportunity of acknowledging my deep sense of gratitude and indebtedness to Professor Sarfaraz Umar for his inspiring guidance and encouragement all along.

I wish to express my sincere thanks to Professor Nisar A. Khan, Head Maths., Z.H. College of Engineering and

Technology, Aligarh Muslim University, Aligarh for his continuous encouragement. I owe a great deal to Professor S. Izhar Husain, Head, Department of Mathematics, Aligarh Muslim University, Aligarh for providing me various facilities during the period of my research in the Department. Finally, I am thankful to my friend Dr. Muneer H. Khan, Lecturer, Department of Mathematics, Aligarh Muslim University, Aligarh without whose cooperation and help, this thesis would have not seen this day.

In the end, I render my thanks to Mr. Fazal Hasnain Naqvi who typed the manuscript with infinite care.

A. Khan

Dated: 11-9-81 (ARMAN KHAN)

ALIGARH.

C E R T I F I C A T E

This is to certify that the contents of this Thesis entitled 'On Certain Problems of Approximation' is an original research work of Mr. Azman Khan done under my supervision.

I further certify that the work in this thesis has not been submitted either partly or fully to any institution for the award of any other degree.

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C O N T E N T S

CHAPTER I

PRELIMINARY AND INTRODUCTION

1.1. The present thesis is based on certain investigations 'ON CERTAIN PROBLEMS OF APPROXIMATION'. Before giving the résumé of the work of earlier researches[1] in the light of which the present work has been done, it seems desirable to state various definitions and notations which the author will require in the sequel.

1.2. Definitions and notations.

Approximations:

Let X be a Banach space of continuous functions on $[a, b]$ with the norm $\|\cdot\|$ defined by $\|f\| = \sup_{x \in [a,b]} |f(x)|$.

Let Φ be a subset of X. An element of X is called approximable by linear combination,

$$P = a_1 \phi_1 + a_2 \phi_2 + \cdots\cdots\cdots a_n \phi_n, \ \phi_i \in \Phi, \ a_i \text{ real},$$

if for each $\epsilon > o$, there is a polynomial P such that

$\| f - P \| < \varepsilon$. If $\bar{\phi} = \{\phi_n\}$ then

(1.2.1) $$E_n^{*}(f) = E_n^{\bar{\phi}}(f) = \inf_{a_1, a_2, \ldots, a_n} \| f - (a_1\phi_1 + a_2\phi_2 + \cdots a_n\phi_n) \|.$$

is called as the n-th degree of approximation of $f(x)$ by the sequence $\{\phi_n\}$. If the infimum (1.2.1) is attained for some P, then this P is called a polynomial (a linear combination) of best approximation.

1.3. Modulus of continuity

Let $\langle a, b\rangle$ denote an interval (which may be either (a, b) or $[a, b]$) and $f(x)$ be defined on $\langle a, b\rangle$. Given a positive number δ, we define the modulus of continuity $w(\delta)$ by

$$w(\delta) = \sup_{|x-y| \le \delta} \left\{ |f(x) - f(y)| \,/\, x, y \in \langle a, b\rangle \right\}$$

A modulus of continuity has the following properties:

(i) $w(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$.

(ii) $w(\delta)$ is positive and increasing.

(iii) w is sub additive, i.e.

$$w(\delta_1 + \delta_2) \le w(\delta_1) + w(\delta_2).$$

(iv) $w(\delta)$ is continuous.

(v) $w(n\delta) \leq nw(\delta)$, if n is natural number.

(vi) $w(\lambda\delta) \leq (\lambda+1)\, w(\delta)$, if λ is any positive number.

1.3.1. Integral modulus of continuity.

The integral modulus of continuity of $f(x)$ is defined as

$$w_p(\delta) = w_p(\delta;\, f) = \sup_{0<|h|\leq\delta} \| f(x+h) - f(x) \|_p \,.$$

where $\| f \|_p = \left[\int_{-\pi}^{\pi} | f(x) |^p dx \right]^{1/p}$, for $f(x)$ is periodic of (period 2π) & $f \in L_p$, $p \geq 1$

1.3.2. Generalised modulus of continuity.

The generalised modulus of continuity of $f(x)$ is defined as

$$w_p^{*}(\delta) = w_p(\delta;\, f) = \sup_{0<|h|\leq\delta} \| f(x+h) + f(x-h) - 2f(x) \|_p \,.$$

1.4. Weierstrass Ist theorem ≠

Each continuous function $f(x)$ on $[a, b]$ is approximable by algebraic polynomials, that is, for each $\epsilon > 0$, there is polynomials $P_n(x)$, defined by

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots\dots\dots + a_n x^n.$$

≠ Weierstrass K. [39]

(where a_i are real coefficients) and an integer $N = N(\varepsilon)$ such that

$$|P_n(x) - f(x)| < \varepsilon \quad \text{for all } n > N, \text{ for all } x \in [a, b]$$

Weierstrass IInd theorem.

Let $f(x) \in C_{2\pi}$. For each $\varepsilon > 0$, there exists a a trignometric polynomial $T(x)$ that for all real x.

$$| T(x) - f(x) | < \varepsilon.$$

Jackson's Ist theorem**

There exists a constant M such that for each $f \in C_{2\pi}$.

$$E_n^*(f) \leq M\,\omega\,(f, \tfrac{1}{n}). \quad , n=1,2 \cdots$$

and even

$$E_n^*(f) \leq M\omega_2\,(f; \tfrac{1}{n}).$$

Bernstein theorem.+

A function $f \in C_{2\pi}$ belongs to the class Lip α, $0 < \alpha < 1$, if and only if

$$E_n^*(f) = O(n^{-\alpha}).$$

** Jackson, D. [11].

+ Bernstein, S.N. [4].

Riemann - Lebesgue theorem.

If $f(x)$ is integrable over (a, b), then

$$\lim_{n\to\infty} \int_a^b f(x) \cos n x \, dx = 0,$$

and

$$\lim_{n\to\infty} \int_a^b f(x) \sin n x \, dx = 0.$$

1.5. Lipschitz Condition:

Let $f(x)$ be a function specified on the interval $\langle a, b\rangle$ and let $0 < \alpha \leq 1$. We say that $f(x)$ satisfies the lipschitz condition with exponent α, if there is a constant M such that,

$$|f(y) - f(x)| \leq M \, |y-x|^{\alpha}, \quad \text{for all } x, y \in \langle a, b\rangle.$$

We shall write this fact briefly as $f(x) \in \mathrm{Lip}_M^{\alpha}$. We shall write Lip_M^{α} merely as $\mathrm{Lip}\,\alpha$ when the constant M is immaterial in our discussion.

We say that $f(x) \in \mathrm{Lip}\ \psi(t)$ if

$$|f(x+t) - f(x)| \leq M\, \psi(t), \ \forall, \ x \in \langle a, b\rangle.$$

where $\psi(t)$ is a positive increasing function.

Lip (α, p) Class[++]

We say that $f(x) \in \text{Lip}\ (\alpha, p)$ class if

$$\left\{ \int_0^{2\pi} |f(x+t) - f(x)|^p \, dx \right\}^{1/p} = O(t^\alpha),\ 0 < \alpha \leq 1,\ p > 1.$$

$\text{Lip}_\alpha\ (J(t))$ Class[+++]

We say that $f(x) \in \text{Lip}\ (J(t))$ class if

$$\left\{ \int_0^{2\pi} |f(x+t) - f(x)|^p \, dx \right\}^{1/p} = O\ (J(t)),\ \ p > 1.$$

where $J(t)$ is a positive increasing function.

$W(L^p, \psi(t))$ Class[£]

We say that $f(x) \in W(L^p, \psi(t))$ Class if

$$\left\{ \int_0^{2\pi} |f(x+t) - f(x)|^p \sin^{\beta p} x \, dx \right\}^{1/p} = O(\psi(t)),\ \text{for}$$

$p > 1$ and $\beta \geq 0$, where $\psi(t)$ is a positive increasing function.

++ Hardy and Little Wood [9]
+++ Siddiqui, A.H. [36]
£ Khan, Huzoor, H. [23]

<u>Class K(t)</u>** A function $f(x)$, integrable L, is said to belong to the class $k(t)$, where $k(t)$ is a positive increasing function and $\frac{k(t)}{t}$ is decreasing such that,

(a) $K(Ax) \leq k(A)\, k(x)$, where A is any constant.

(b) $|\, f(x+t) - f(x)| = O\,(k(t))$.

<u>Class h(t)</u>** A function $f(x)$ is said to belong to the class $h(t)$ if $f(x) \in L_1\,(-\infty, +\infty)$ and

(a) $h(\alpha x) = h(\alpha)\, h(x)$, α is any constant.

(b) $\int_0^u [f(x+t) + f(x-t) - 2\, f(x)]\, dt = O(h(u))$, $u \to 0$.

<u>Class g(t)</u>** A function $f(x)$ is said to belong to the class $g(t)$ if $f(x) \in L_1(-\infty, +\infty)$ and

(a) $g(\alpha x) = g(\alpha)\, g(x)$, α is any constant

(b) $\int_0^u \phi(t)\, dt = O\,(g(u))$, where

$$\phi(t) = \int_{-\infty}^{\infty} |f(x+t) + f(x-t) - 2f(x)\,|\, dx$$

** Khan, A. etal [14].

** Khan, Arman and S. Umar [16],[17]

1.6. Singular integrals used in the text.

(a) Fejér singular integral.

(1.6.1)
$$\sigma_n(t) = \frac{1}{2\pi(n+1)} \int_{-\pi}^{\pi} f(x+t)\left[\frac{\sin(n+1)\, t/2}{\sin t/2}\right]^2 dt$$

$$(n \to \infty).$$

(b) Abel- Poisson singular integral.

$$\sigma(r, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t)\, \frac{1-r^2}{1-2r \cos t + r^2}\, dt \quad (r \to 1-)$$

(c) De La Vallée - Poussin.

$$V_n(t) = \frac{(2n)(2n-2)\ldots\ldots 2}{(2n-1)(2n-3)\ldots 3.1}\; \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \cos^{2n} \frac{t}{2}\, dt$$

$$(n \to \infty).$$

(d) Poisson - Cauchy singular integral.

$$P(x, \xi) = \frac{\xi}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t)}{t^2+\xi^2}\, dt, \quad \xi \to 0+.$$

(e) <u>Gauss - Weierstrass singular integral.</u>

$$w(x;\xi) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} f(x+t)\; e^{-t^2/\xi}\; dt, \quad \xi \to 0+.$$

or

$$W(x;\xi) = \sqrt{\frac{1}{\xi}} \int_{-\pi}^{\pi} f(x+t)\, e^{-t^2/\xi}\, dt, \quad \xi \to 0+$$

(f) <u>Jackson - De La Vallée - Poussin singular integral.</u>

$$J_{2n}(t) = \frac{3}{2\pi(n+1)(2n^2+4n+3)} \int_{-\pi}^{\pi} f(x+t)\left[\frac{\sin(n+1)\,t/2}{\sin t/2}\right]^4 dt$$

$$(n \to \infty).$$

(g) <u>Picard singular integral.</u>

$$P(x;\xi) = \frac{1}{2\xi} \int_{-\infty}^{\infty} f(x+t)\; e^{-|t|/\xi}\; dt, \quad \xi \to 0+.$$

(h) <u>Generalised Gauss - Weierstrass singular integral.</u>

$$L(x;\xi;n) = \frac{1}{2\sqrt{n+1/n}}\;\frac{1}{\xi^{1/n}} \int_{-\infty}^{\infty} f(x+t)\; e^{-|t|^n/\xi}\; dt, \quad \xi \to 0$$

(h) <u>[illegible]'s operator</u>

$$A(x;\xi) = \frac{1}{2\,\xi^{2}\sqrt{\xi}} \int_{-\infty}^{\infty} f(x+t)\; e^{-(|t|/\xi)^{1/\xi}}, \quad \xi \to 0+.$$

1.7. Formulas** used in the text.

(1.7.1) $\overset{*}{\gamma}(a, x) = \frac{x^{-a}}{\Gamma a} \int_0^x e^{-t} t^{a-1} \, dt.$

(1.7.2) $\overset{*}{\gamma}(-n, x) = x^n.$

(1.7.3) $\Gamma(\alpha, \beta) = \Gamma\alpha - \gamma(\alpha, \beta) = \int_\beta^\infty x^{\alpha-1} e^{-x} \, dx.$

(1.7.4) $\gamma(\alpha, \beta) = \int_0^\beta x^{\alpha-1} e^{-x} \, dx.$

(1.7.5) $\overset{*}{\gamma}(\alpha, \beta) = \frac{\beta^{-\alpha}}{\Gamma\alpha} \gamma(\alpha, \beta).$

(1.7.6) In complete Gamma function

$$P(a, x) = \frac{1}{\Gamma a} \int_0^x e^{-t} t^{a-1} \, dt.$$

(1.7.7) $\overset{*}{\gamma}(a, x) = x^{-a} P(a, x)$

(1.7.8) $\Gamma 2\xi / \Gamma\xi = \frac{1}{2}$

1.8. Gamma function : We define the gamma function.

** Abramowitz and Segun [1].

$\Gamma(n)$, for $n > 0$, by the relation

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt.$$

1.9. Let $f(x)$ be a periodic function of period 2π and integrable in the sense of Lebesgue $(-\pi, \pi)$. Let the Fourier series of $f(x)$ be given by

$$f(x) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \operatorname{Cos} kx + b_k \operatorname{Sin} k x)$$

$$= \sum_{k=0}^{\infty} A_k(x).$$

1.10. <u>Notations.</u>

We write

(1.10.1) $\phi(t) = f(x+t) + f(x-t) - 2f(x).$

(1.10.2) $\phi_x(t) = \frac{1}{2}\left\{f(x+t) + f(x-t) - 2f(x)\right\}$

(1.10.3) $\Phi(t) = \int_0^t \phi_x(v)\, dv.$

(1.10.4) $\lambda(t) = \int_{-\infty}^{\infty} |f(x+t) + f(x-t) - 2f(x)|\, dx$

Introduction

1.11. The problem of determining the degree of approximation of the functions belonging to different classes (classes of continuous functions, modulus of continuity, Lip α and Lip$(\psi(t), p)$) by using the finite summability operators (Cesaro, Abel, Nörlund, generalised Nörlund, Triangular and Matrix operators) have been obtained by Alexit, G. [3], Chandra, P. [7], Flett, T.M. [8], Holland, A.S.B. et al [10], Kathal, P.D. et al [12], Khan, Huzoor H. and S. Umar [24,25,26,27,28,30], Sahney, B.N. et al [35] and Siddiqui A.H. [37]. So for infinite summability operators have not been used in this problem, therefore in this case, it reduces essentially to the convergence problem.

Now a natural question arises, can one obtain the same degree of approximation under some restriction on the functions involved ? In relation, however, to infinite summability operators the corresponding problem has been studied only by Khan Huzoor H. and S. Umar [32] by imposing certain conditions on the functions involved for Borel and (J, p_n)

operators using Lip ($\psi(t)$, p) class.

Later on, Khan, Huzoor H. and S. Umar [29] obtained the degree of approximation of functions belonging to the class Lip ($\psi(t)$, p) using A_λ – operators. (of course, an infinite operator) without imposing any conditions on the function involved.

1.12. Considering the idea used by the mathematician for obtaining the degree of approximation for summability operators, we have tried the same problem for the virgin singular integrals (1.3. (d), (e), (g), (h) and (i)) for the newly defined classes k(t), h(t) and g(t). It may be remarked that the operators which we have used for obtaining the degree of approximation ultimately have almost the same order of approximation.

1.13. Butzer, P.L. [5] determined the degree of approximation and obtained a better result, on considering the functions belonging to the generalised modulus of continuity $w(\delta)$ of the class of all real valued continuous periodic functions on the whole real axis, $f(x)$ and the singular integral of De La vallée-Poussin.

His theorem states as follows:

Theorem (A) : If $f(x) \in C_{2\pi}$ has a generalised modulus continuity $\overset{*}{w}(\delta)$, then for all x

$$| V_n(x) - f(x) | \leq \frac{3}{2} \overset{*}{w} (\frac{1}{\sqrt{n}})$$

where $V_n(x)$ is the De La vallée-Poussin singular integral.

In chapter II, we have for the first time determined the degree of approximation to a function belonging to the class $k(t)$ by Gauss-weierstrass and Generalised Gauss - weierstrass singular integrals.

Our theorems state as follows:

Theorem (B): For the functions $f(x) \in k(t)$

$$|W(x; \xi) - f(x)| = O (\xi^{1/2} k(\xi)), \quad \xi \to 0 + .$$

Provided $k(t)$ satisfies the following conditions:

$$\int_{\sqrt{\xi}}^{\pi/\sqrt{\xi}} \frac{k(t)}{t^2}\, dt = O\left(\frac{k(\sqrt{\xi})}{\xi}\right).$$

where $W(x;\xi)$ is the Gauss-weierstrass singular integral.

Theorem (C): If $f(x) \in k(t)$, then

$$| L(x;\xi;s) - f(x) | = O\left(\xi^{1-1/s}\, k(\xi)\right), \quad \xi \to 0+.$$

where $L(x;\xi;s)$ is generalised Guass-weierstrass singular integral.

It may be remarked that our result is more general because the class $k(t)$ reduces to modulus of continuity in particular case $p = 1$.

1.14. Butzer, P.L. [6] obtained the degree of approximation for singular integrals (Abel-Poisson and Poisson-Cauchy) for the functions belonging to the class Lip α.

His theorems state as follows:

Theorem (D): If $f(x) \in L_p(-\pi, +\pi)$, $1 \leq p < +\infty$ and also belong to $C[-\pi, +\pi]$ and $\| \sigma(r, x) - f(x) \| = O(1-r)$

then $f(x)$ is constant, where $\sigma(r, x)$ is Abel-Poisson singular integral.

<u>Theorem</u> (E): Let $f(x) \in L_p(-\infty, +\infty)$, $1 < p < +\infty$

(a) If $\lim\limits_{\xi \to 0} \frac{1}{\xi} \| f(x; \xi) - f(x) \| = 0$, then $f(x) = 0$

(b) If $\tilde{f}'(x) \in L_p(-\infty, +\infty)$, then

$$\| f(x; \xi) - f(x) \| \leq \xi \| \tilde{f}'(x) \| .$$

(c) If $\|f(x; \xi) - f(x)\| = O(\xi)$, then $\tilde{f}'(x) \in L_p(-\infty, +\infty)$

where $f(x; \xi)$ is Poisson-Cauchy singular integral.

In the chapter III, we have tried the same problem as in the chapter II for singular integrals (Picard, Poisson-Cauchy and ostrowski's operator) and succeeded in obtaining a better approximation generalising the results of Butzer, P.L. [6].

Our results are as follows:

<u>Theorem</u> (F) : If $f(x) \in k(t)$, then as $\xi \to 0+$,

(i) $|P(x;\xi) - f(x)| = O(k(\xi))$.

(ii) $|\overset{*}{P}(x;\xi) - f(x)| = O(k(\xi))$.

(iii) $|A(x;\xi) - f(x)| = O(k(\xi))$.

where $P(x;\xi)$, $\overset{*}{P}(x;\xi)$ and $A(x;\xi)$ are Picard, Poisson-Cauchy and Ostrowski's operators.

1.15. Butzer, P.L. [5] proceeded to discuss the problem of approximation for functions $f(x) \in L_p$ having an integral modulus of continuity or belonging to the class Lip $(\alpha; p)$ for $0 < \alpha \leq 1$, $p \geq 1$.

He studied the approximation in the mean of $f(x)$ by De La Vallée-Poisson singular integral.

If $f(x)$ is periodic (period 2π) and $f \in L_p$, $p \geq 1$ with

$$\| f \|_p = \left[\int_{-\pi}^{\pi} | f(x)|^p \, dx \right]^{1/p}$$

It is clear that $w_p^*(\delta) \leq 2\, w_p(\delta)$. If $w_p(\delta) \leq M\, \delta^\alpha$, $0 < \alpha \leq 1$, then $f(x)$ is said to satisfy an integral lipshitz condition of $f \in Lip(\alpha; p)$. Khan, Huzoor H. [22] generalised the integral modulus of continuity class as,

$$w^{*}(\delta; p; \beta) = \sup_{o<|h|\leq\delta} \| [f(x+h) - f(x)] \sin^{\beta} t \|, \text{ for}$$

$\beta \geq o$. It can be reduced that for $\beta = o$, the class defined above reduces to $w_p^{*}(\delta)$.

It has been shown by Quade [34] that the class of functions satisfying the condition $Lip(\alpha; p)$, $o < \alpha < 1$ is identical with the class of functions approximable in the mean pth power with error $O(\frac{1}{n^{\alpha}})$ by trigonometric polynomials of order n. For the class $\overset{*}{w}(\delta, p, \beta)$ the above remark of Quade [34] is also true.

For $\alpha = 1$, Zygmund [40] has shown that this class is equivalent to the class of functions approximable in the mean pth power with error $O(\frac{1}{n})$ by trigonometric polynomials of order n.

Butzer, P.L. [5] proved the following theorems:

<u>Theorem</u> (G) : If $f(x)$ has generalised modulus of continuity $w_p^*(\delta)$, $p \geq 1$, then

$$\| V_n(x) - f \|_p = O\left(w_p^*\left(\frac{1}{n^2}\right)\right).$$

where $V_n(x)$ is a De La Vallée-Poussin singular integral.

He further extended his result for Gauss-weierstrass singular integral as follows:

<u>Theorem</u> (H) : If $f(x) \in L_p(-\infty, +\infty)$, $1 < p < \infty$ and

$$\| W(x; \xi) - f(x) \| = O(\xi), \text{ then}$$

$f(x)$, $f'(x)$ and $f''(x)$ belong to $L_p(-\infty, +\infty)$.

In chapter IV, we have determined the degree of approximation of functions belonging to the classes $h(t)$ and $g(t)$ by Gauss-weierstrass singular integral generalising the result of Butzer, P.L. [6].

Our theorems state as follows:

<u>Theorem</u> (I) : If $f(x)$ belongs to the class $h(t)$, then

$$|W(x;\xi) - f(x)| = O\left(\frac{h(\sqrt{\xi})}{\sqrt{\xi}}\right), \quad \xi \to o+.$$

provided $h(t)$ satisfies the following conditions:

$$\frac{h(t_o)}{h(\sqrt{\xi})} \quad e^{-(t_o/\sqrt{\xi})^2} = o\,(1),$$

$$\frac{d}{dt}\,(h(t)) = O\left(\frac{h(t)}{t}\right), \text{ and}$$

$$\int_{t_o/\sqrt{\xi}}^{\infty} \frac{h(\sqrt{\xi}\,v)}{v} \quad e^{-v^2}\,dv < +\infty$$

where, we break the interval (o, ∞) into (o, t_o) and (t_o, ∞).

<u>Theorem</u> (J) : If $f(x)$ belongs to the class $g(t)$, then

$$\int_{-\infty}^{\infty} |W(x;\xi) - f(x)|\,dx = O\left(\frac{g(\sqrt{\xi})}{\sqrt{\xi}}\right), \quad \xi \to o+.$$

provided $g(t)$ satisfies the following conditions

$$\frac{g(t_o)}{g(\sqrt{\xi})} \quad e^{-(t_o/\sqrt{\xi})^2} = o\,(1)$$

$$\frac{d}{dt}\,(g(t)) = O\left(\frac{g(t)}{t}\right), \text{ and}$$

$$\int_{t_0/\sqrt{\xi}}^{\infty} \frac{g(v)}{v} e^{-v^2} dv < +\infty .$$

1.16. Using the conditions $O(\frac{h(t)}{t})$ and $O(\frac{g(t)}{t})$ on $\phi_x(t)$ in place of $O(h(t))$ and $O(g(t))$ respectively, we have the following theorems:

Theorem(K) : If $f(x) \in h(t)$, then

$$|W(x;\xi) - f(x)| = O\left(\frac{h(\sqrt{\xi})}{\xi}\right), \quad \xi \to 0+ .$$

provided that,

$$\left(\frac{\sqrt{\xi}}{t_0}\right) \frac{h(t_0)}{h(\sqrt{\xi})} e^{-(t_0/\sqrt{\xi})^2} = o(1),$$

$$\frac{d}{dt}\left(\frac{h(t)}{t}\right) = O\left(\frac{h(t)}{t^2}\right), \text{ and}$$

$$\int_{t_0/\sqrt{\xi}}^{\infty} \frac{h(v)}{v^2} e^{-v^2} dv < +\infty .$$

Theorem (L) : If $f(x) \in g(t)$, then

$$\int_{-\infty}^{\infty} | W(x;\xi) - f(x) | dx = O\left(\frac{g(\sqrt{\xi})}{\xi}\right) .$$

under the conditions:

$$\left(\frac{\sqrt{\xi}}{t_0}\right) \frac{g(t_0)}{g(\sqrt{\xi})} \; e^{-(t_0/\sqrt{\xi})^2} = o\,(1)$$

$$\frac{d}{dt}\left(\frac{g(t)}{t}\right) = O\left(\frac{g(t)}{t^2}\right), \text{ and}$$

$$\int_{t_0/\sqrt{\xi}}^{\infty} \frac{g(v)}{v^2} \; e^{-v^2} \; dv < +\infty .$$

1.17. In chapter Vth, we have extended our results of previous chapter by using more general operator (Generalised Gauss-weierstrass singular integral) for the same classes and determined the degree of approximation.

It may be remarked that all the results of chapter IVth are reduced by our theorems of chapter Vth by taking $s = \frac{1}{2}$.

In fact, we state our theorems as follows:

Theorem (X) : If $f(x)$ belongs to the class $h(t)$ then

$$|L(x; \xi; s) - f(x)| = O\left(\frac{h(\xi^{1/s})}{\xi^{1/s}}\right), \quad \xi \to 0+ .$$

provided that

$$\frac{h(t_o)}{h(\xi^{1/s})} \; e^{-|t_o|^s/\xi} = o(1),$$

$$\frac{d}{dt}(h(t)) = O\left(\frac{h(t)}{t}\right), \quad \text{and}$$

$$\int_{t_o/\xi^{1/s}}^{\infty} \frac{h(v)}{v} \; e^{-v^s} dv < +\infty$$

<u>Theorem</u> (H) : If $f(x) \in g(t)$, then

$$\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)| \, dx = O\left(\frac{g(\xi^{1/s})}{\xi^{1/s}}\right)$$

under the conditions,

$$\frac{g(t_o)}{g(\xi^{1/s})} \; e^{-|t_o|^s/\xi} = o(1),$$

$$\frac{d}{dt}(g(t)) = O\left(\frac{g(t)}{t}\right), \quad \text{and}$$

$$\int_{t_o/\xi^{1/s}}^{\infty} \frac{g(v)}{v} \; e^{-v^s} dv < +\infty .$$

Further, we have considered the conditions $O\left(\frac{h(t)}{t}\right)$ and $O\left(\frac{g(t)}{t}\right)$ on $\phi_x(t)$ instead of $O(h(t))$ and $O(g(t))$ respectively, we have the following theorems:

<u>Theorem</u> (O) : If $f(x) \in h(t)$, then

$$|L(x;\ \xi;s) - f(x)| = O\left(\frac{h(\xi^{1/s})}{\xi^{2/s}}\right)$$

provided that,

$$\left(\frac{\xi^{1/s}}{t_0}\right)\ \frac{h(t_0)}{h(\xi^{1/s})}\ e^{-|t_0|^s/\xi} = o(1),$$

$$\frac{d}{dt}\left(\frac{h(t)}{t}\right) = O\left(\frac{h(t)}{t^2}\right), \text{ and}$$

$$\int_{t_0/\xi^{1/s}}^{\infty} \frac{h(v)}{v^2}\, e^{-v^s}\, dv < +\infty$$

<u>Theorem</u> (P) : If $f(x) \in g(t)$, then

$$\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)|\, dx = O\left(\frac{g(\xi^{1/s})}{\xi^{2/s}}\right)$$

under the conditions,

$$\left(\frac{\xi^{1/s}}{t_0}\right)\frac{g(t_0)}{g(\xi^{1/s})}\ e^{-|t_0|^s/\xi} = o(1)$$

$$\frac{d}{dt}\left(\frac{g(t)}{t}\right) = O\left(\frac{g(t)}{t^2}\right), \text{ and}$$

$$\int_{t_0 \xi^{1/a}}^{\infty} \frac{g(v)}{v^2}\, e^{-v^2}\, dv < +\infty .$$

1.18. Numerous workers have already used the singular integrals for obtaining degree of approximation belonging to different classes. We have also used the same for determining the degree of approximation.

It is well known that if $0 < \alpha < 1$, the necessary and sufficient condition for the periodic function $f(x)$ to belong to the class Lip α is that,

$|f(x) - \sigma_n(x)| = O\left(\frac{1}{n^\alpha}\right)$ uniformly, where $\sigma_n(x)$ is the Fejér sum of order n of the Fourier series of $f(x)$. For $\alpha = 1$, the result is no longer true, but Alexits, G.[2] determined that $f \in$ Lip 1 iff,

$|\tilde{f} - \tilde{\sigma}_n| = O\left(\frac{1}{n}\right)$, where $\tilde{f}$ is the conjugate of f and $\tilde{\sigma}_n$ is its Fejér sum. His main result is a unification and generalisation of these results in the following theorems:

<u>Theorem</u> (Q) : Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$, then $f \in \mathrm{Lip}\,(\alpha; p)$ iff

$\| \tilde{f} - \tilde{\sigma}_n \|_p = O\left(\frac{1}{n^\alpha}\right)$, where $\| \quad \|_p$ denotes the norm in the metric L^p.

In chapter VI, we have used the singular integrals (Picard and Ostrowski) only for obtaining the degree of approximation of functions belonging to the classes $H(t)$ and $g(t)$.

Our theorems for Picard singular integral are as follows:

<u>Theorem</u> (R) : If $f(x) \in h(t)$ then

$$|P(x; \xi) - f(x)| = O\left(\frac{h(\xi)}{\xi}\right), \quad \xi \to 0+.$$

provided $h(t)$ satisfies the following:

$$\frac{h(t_0)}{h(\xi)} \quad e^{|t_0|/\xi} \quad = o\ (1)$$

$$\frac{d}{dt}(h(t)) = O\left(\frac{h(t)}{t}\right), \quad \text{and}$$

$$\int_{t_0/\xi}^{\infty} \frac{h(v)}{v} e^{-v}\, dv < +\infty .$$

__Theorem__ (S): If $f(x) \in g(t)$, then

$$\int_{-\infty}^{\infty} |F(x;\xi) - f(x)|\, dx = O\left(\frac{g(\xi)}{\xi}\right), \quad \xi \to 0+ .$$

under the following conditions:

$$\frac{g(t_0)}{g(\xi)} e^{-|t_0|/\xi} = o\,(1),$$

$$\frac{d}{dt}(g(t)) = O\left(\frac{g(t)}{t}\right)$$

$$\int_{t_0/\xi}^{\infty} \frac{g(v)}{v} e^{-v}\, dv < +\infty$$

__1.19.__ We have the following theorems for ostrowski's operator

__Theorem__ (T) : If $f(x) \in h(t)$, then

$$|A(x;\xi) - f(x)| = O\left(\frac{h(\xi)}{\xi^2}\right)$$

under the following conditions:

$$\frac{h(t_0)}{h(\xi)} \; e^{-(|t_0|/\xi)^{1/\xi}} = o(1),$$

$$\frac{d}{dt}(h(t)) = O\left(\frac{h(t)}{t}\right), \text{ and}$$

$$\int_{t_0/\xi}^{\infty} \frac{H(v)}{v} \; e^{-v^{1/\xi}} \; dv < +\infty$$

__Theorem__ (U) : If $f(x) \in g(t)$, then

$$\int_{-\infty}^{\infty} |A(x;\xi) - f(x)| \, dx = O\left(\frac{g(\xi)}{\xi^2}\right)$$

provided $g(t)$ satisfies the following:

$$\frac{g(t_0)}{g(\xi)} \; e^{-(|t_0|/\xi)^{1/\xi}} = o(1),$$

$$\frac{d}{dt}(g(t)) = O\left(\frac{g(t)}{t}\right), \text{ and}$$

$$\int_{t_0/\xi}^{\infty} \frac{g(v)}{v} \; e^{-v^{1/\xi}} \; dv < +\infty$$

Where, we break the interval $(0, \infty)$ into $(0, t_0)$ and (t_0, ∞).

CHAPTER II

ON THE DEGREE OF APPROXIMATION TO A FUNCTION BELONGING TO THE CLASS $k(t)$

2.1 Let $f(x)$ be a Lebesgue integrable function with period 2π and let

$$(2.1.1) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$= \sum_{k=0}^{\infty} A_k(x)$$

be its Fourier series.

Considering the functions belonging to the generalised modulus of continuity $w^{*}(\delta)$ of the class of all real valued continuous periodic functions on the whole real axis, $f(x)$ and the singular integral of De la Vallee-Poissin, Butzer, P.L. [5] determined the degree of approximation of functions and obtained a better result under certain conditions.

His theorem states as follows.

<u>Theorem</u> (2.1.1). If $f(x) \in C_{2\pi}$ has a generalised modulus of continuity $w^{*}(\delta)$, then for all x

(2.1.2) $$|V_n(x) - f(x)| \leq \frac{3}{2} w^*\left(\frac{1}{\sqrt{n}}\right).$$

where $V_n(x)$ is the De la Vallée-Poussin operator.

2.2 In this section of the Chapter, we have for the first time determined the degree of approximation to a function by Gauss-Weierstrass singular integrals of the Fourier series of $f(x)$ belonging to the class $k(t)$, where $k(t)$ is a positive increasing function and $k(t)/t$ is decreasing such that,

(a) $k(A_x) \leq k(A)\, k(x)$, where A is constant.

(b) $|f(x+t) - f(x)| = O(k(t))$, for $f(x) \in k(t)$.

We notice that

(i) If $k(t) = t^{\alpha}$, then our class reduces to Lip α

(ii) If $k(t) = t^{\alpha - 1/p}$ and $f(x) \in L^p$ $(p \geq 1)$, then our class reduces to $\mathrm{Lip}(\alpha, p)$*.

(iii) If $k(t) = \psi(t)\, t^{-1/p}$, where $\psi(t)$ is a positive increasing function s.t. $\psi(Ax) \leq \psi(A)\psi(x)$ and $f(x) \in L^p$ $(p \geq 1)$, then our class reduces to $\mathrm{Lip}(\psi(t), p)$⁺ for $p > 1$.

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* Hardy and Littlewood [9]

⁺ Khan, Huzoor; H. [21]

It may be remarked that our result is more general because the class $k(t)$ reduces to modulus of continuity in particular case $p = 1$.

Our theorem states as follows.

<u>Theorem</u>* (2.2.1). For the functions $f(x) \in k(t)$,

$$|W(x\,;\,\xi) - f(x)| = O(\xi^{1/2}\, k(\xi)),$$

provided $k(t)$ satisfies the following.

$$\int_{\sqrt{\xi}}^{\pi/\sqrt{\xi}} \frac{k(t)}{t^2}\, dt = O\left(\frac{k(\sqrt{\xi})}{\sqrt{\xi}}\right), \quad \xi \to 0+.$$

<u>Proof of theorem</u> (2.2.1)

We know that

$$W(x\,;\,\xi) - f(x) = \sqrt{\frac{1}{\xi}} \int_{-\pi}^{\pi} [f(x)+f(x-t)-2f(x)]\, e^{-t^2/\xi}\, dt + R(x\,;\,\xi).$$

$$= 2\sqrt{\frac{1}{\xi}} \int_{0}^{\pi} \phi(t)\, e^{-t^2/\xi}\, dt + o(\xi)^{\phi}$$

$$= 2\sqrt{\frac{1}{\xi}} \left[\int_{0}^{\xi} + \int_{\xi}^{\pi}\right] \phi(t)\, e^{-t^2/\xi}\, dt.$$

where $o(\xi) \to 0$ as $\xi \to 0$.

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* Khan, A. et al [14].

ϕ Sunouchi and Watari [38]

$$= I_1 + I_2, \text{ say}.$$

For evaluating I_1, we have

$$I_1 = 2\sqrt{\frac{\pi}{\xi}} \int_0^{\xi} \phi(t)\, e^{-t^2/\xi}\, dt.$$

$$|I_1| \leq 2\sqrt{\frac{\pi}{\xi}} \int_0^{\xi} |\phi(t)|\, e^{-t^2/\xi}\, dt$$

$$\leq 2\sqrt{\frac{\pi}{\xi}} \int_0^{\xi} k(t)\, e^{-t^2/\xi}\, dt\,,$$

$$= 2\sqrt{\frac{\pi}{\xi}}\, k(\xi) \int_0^{\xi} e^{-t^2/\xi}\, dt.$$

putting $t/\sqrt{\xi} = u$, we get

$$= O(k(\xi)) \int_0^{\sqrt{\xi}} e^{-u^2}\, du\,.$$

$$I_1 = O(k(\xi))\, O(\xi^{1/2}). \quad (\text{as } e^{-u^2} \text{ decreasing})$$

$$= O(\xi^{1/2}\, k(\xi)).$$

Again

$$I_2 = 2\sqrt{\frac{\pi}{\xi}} \int_{\xi}^{x} \phi(t)\, e^{-t^2/\xi}\, dt.$$

$$|I_2| \leq 2\sqrt{\frac{\pi}{\xi}} \int_{\xi}^{x} |\phi(t)|\, e^{-t^2/\xi}\, dt$$

$$= 2\sqrt{\frac{\pi}{\xi}} \int_{\xi}^{x} k(t)\, e^{-t^2/\xi}\, dt.$$

putting $t = \sqrt{\xi}\, u$, we have

$$= O(k(\xi^{1/2})) \int_{\sqrt{\xi}}^{\pi/\sqrt{\xi}} e^{-u^2} k(u)\, du .$$

$$= O(k(\xi^{1/2})) \int_{\sqrt{\xi}}^{\pi/\sqrt{\xi}} u^{-2} u^2 e^{-u^2} k(u)\, du$$

$$= O(k(\xi^{1/2}))\, o(\xi) \int_{\sqrt{\xi}}^{\pi/\sqrt{\xi}} u^{-2} k(u)\, du .$$

$$= O(k(\xi^{1/2}))\, O(\xi)\, O\left(\frac{k(\xi^{1/2})}{\sqrt{\xi}} \right)$$

Hence

$$I_2 = O(\xi^{1/2} k(\xi)).$$

Adding the bounds for I_1 and I_2, we have

$$|W(x ; \xi) - f(x)| = O(\xi^{1/2} k(\xi)).$$

This completes the proof of the theorem (2.2.1).

2.3. It can be easily seen that giving different values to $k(t)$, we get the following Corollaries of the above theorem.

Corollary (2.3.1). If $k(t) = t^{\alpha}$, we have

$$|W (x ; \xi) - f(x) | = O(\xi^{\alpha+\frac{1}{2}}).$$

<u>Lemma</u> (2.4.1) If $s > 0$, then

$$\frac{1}{\Gamma\frac{1}{s}} \int_0^1 x^{\frac{1}{s}-1} e^{-x}\, dx = 1$$

<u>Lemma</u> (2.4.2). If $s > 0$, then

$$\frac{1}{\Gamma\frac{1}{s}} \int_1^\infty x^{\frac{2}{s}-1} e^{-x}\, dx = 0.$$

<u>Proof of Lemma</u> (2.4.1)

Using (1.7.1) and (1.7.2), we have

$$\gamma^*\left(\frac{1}{s}, 1\right) = 1$$

therefore,

$$\frac{1}{\Gamma\frac{1}{s}} \int_0^1 e^{-x} x^{\frac{1}{s}-1}\, dx = 1$$

which completes the proof of lemma (2.4.1).

<u>Proof of Lemma</u> (2.4.2)

Using (1. .3), we have

$$\Gamma\left(\frac{2}{s}, 1\right) = \Gamma\frac{2}{s} - \gamma\left(\frac{2}{s}, 1\right)$$

(2.4.1) $$= \int_1^\infty x^{\frac{2}{s}-1} e^{-x}\, dx$$

Now, using (1.7.4), we see that

$$\frac{1}{\Gamma\frac{1}{s}}\int_{1}^{\infty} x^{\frac{2}{s}-1}\, e^{-x}\, dx = \frac{1}{\Gamma\frac{1}{s}}\ \Gamma\left(\frac{2}{s}, 1\right)$$

$$= \frac{\Gamma\frac{2}{s} - \gamma\left(\frac{2}{s}, 1\right)}{\Gamma\frac{1}{s}}$$

$$= \frac{\Gamma\frac{2}{s}}{\Gamma\frac{1}{s}} - \frac{\Gamma\frac{2}{s}}{\Gamma\frac{1}{s}} \cdot \frac{\gamma\left(\frac{2}{s}, 1\right)}{\Gamma\frac{2}{s}}$$

$$= \frac{\Gamma\frac{2}{s}}{\Gamma\frac{1}{s}}\left[1 - \frac{\gamma\left(\frac{2}{s}, 1\right)}{\Gamma\frac{2}{s}}\right]$$

$$= \frac{\Gamma\frac{2}{s}}{\Gamma\frac{1}{s}}\left[1 - \gamma^{*}\left(\frac{2}{s}, 1\right)\right], \text{ by } (2.4.2)$$

$$= 0 \text{ as } \gamma^{*}\left(\frac{2}{s}, 1\right) = 1.$$

Hence

$$\frac{1}{\Gamma\frac{1}{s}}\int_{1}^{\infty} x^{\frac{2}{s}-1}\, e^{-x}\, dx = 0.$$

This completes the proof of Lemma (2.4.2).

<u>Proof of theorem</u> (2.4.1)

We know that

$$L(x;\ \xi;\ s) - f(x) = \frac{s}{2\Gamma\left(\frac{1}{s}\right)}\,\frac{1}{\xi^{1/s}}\int_{-\infty}^{\infty}\Big[f(x+t) + f(x-t) - 2f(x)\Big]\, e^{-|t|^{s}/\xi}\,dt + R(x;\ \xi).$$

$$= \frac{s}{\Gamma\left(\frac{1}{s}\right)\xi^{1/s}}\int_{0}^{\infty}\phi(t)\, e^{-|t|^{s}/\xi}\,dt + o(\xi).^{\neq}$$

$$= \frac{s}{\Gamma\left(\frac{1}{s}\right)\xi^{1/s}}\Big[\int_{0}^{\xi} + \int_{\xi}^{\infty}\Big]\phi(t)\, e^{-|t|^{s}/\xi}\,dt,$$

where $o(\xi) \to 0$, as $\xi \to 0$.

$$= I_1 + I_2, \text{ say.}$$

Evaluating I_1, we see that

$$I_1 = \frac{s}{\Gamma\left(\frac{1}{s}\right)\xi^{1/s}}\int_{0}^{\xi}\phi(t)\, e^{-|t|^{s}/\xi}\,dt.$$

$$|I_1| \leq \frac{s}{\Gamma\left(\frac{1}{s}\right)\xi^{1/s}}\int_{0}^{\xi}|\phi(t)|\, e^{-|t|^{s}/\xi}\,dt.$$

$\neq$ Sunouchi and Watari [38]

$$= \frac{s}{\Gamma\frac{\lambda}{s}\,\xi^{\lambda/s}} \int_0^\xi k(t)\, e^{-|t|^s/\xi}\,dt.$$

$$= \frac{s}{\Gamma\frac{\lambda}{s}\,\xi^{\lambda/s}}\, k(\xi) \int_0^\xi e^{-|t|^s/\xi}\,dt.$$

Now, putting $t = \xi v$, we have

$$= \frac{s}{\Gamma\frac{\lambda}{s}}\, k(\xi)\ \xi^{1-\frac{\lambda}{s}} \int_0^1 e^{-v^s}\,dv .$$

$$= O(\xi^{1-\frac{\lambda}{s}})\, O(k(\xi)).\ \frac{1}{\Gamma\frac{\lambda}{s}} \int_0^1 x^{\frac{\lambda}{s}-1}\, e^{-x}\,dx,$$

on putting $v^s = x$

$$= O(\xi^{1-\frac{\lambda}{s}}\, k(\xi))\ .\ O(1)$$

$$= O(\xi^{1-\frac{\lambda}{s}}\, k(\xi)).$$

Again

$$I_2 = \frac{s}{\Gamma\frac{\lambda}{s}\,\xi^{\lambda/s}} \int_\xi^\infty \phi(t)\, e^{-|t|^s/\xi}\,dt$$

$$|I_2| \leq \frac{s}{\Gamma\frac{1}{s}\,\zeta^{1/s}} \int_{\zeta}^{\infty} |\phi(t)|\; e^{-|t|^{s}/\zeta}\, dt .$$

$$= \frac{s}{\Gamma\frac{1}{s}\,\zeta^{1/s}} \int_{\zeta}^{\infty} k(t)\; e^{-|t|^{s}/\zeta}\, dt .$$

$$= \frac{s}{\Gamma\frac{1}{s}\,\zeta^{1/s}} \cdot \frac{k(\zeta)}{\zeta} \int_{\zeta}^{\infty} t\; e^{-|t|^{s}/\zeta}\, dt .$$

Putting $t = \zeta v$, we have

$$= \frac{s}{\Gamma\frac{1}{s}}\, \zeta^{1-\frac{1}{s}}\, k(\zeta) \int_{1}^{\infty} v\, e^{-v^{s}}\, dv .$$

$$= O(\zeta^{1-\frac{1}{s}}\, k(\zeta))\, \frac{1}{\Gamma\frac{1}{s}} \int_{1}^{\infty} x^{\frac{2}{s}-1}\, e^{-x}\, dx,$$

on putting $v^{s} = x$

$= 0$

Adding the bounds, we see that

$$I_1 + I_2 = O(\zeta^{1-\frac{1}{s}}\, k(\zeta)).$$

Hence

$$\left|L(x;\ \xi;\ s) - f(x)\right| = O\left(\xi^{1-\frac{1}{s}} k(\xi)\right).$$

2.5. The following are the Corollaries of our result (2.4.1).

<u>Corollary</u> (2.5.1). If $k(t) = t^{\alpha}$, we have

$$\left|L(x;\ \xi;\ s) - f(x)\right| = O\left(\xi^{\alpha+1-\frac{1}{s}}\right).$$

<u>Corollary</u> (2.5.2). If $k(t) = t^{\alpha-\frac{1}{p}}$, we get

$$\left|L(x;\ \xi;\ s) - f(x)\right| = O\left(\xi^{\alpha-\frac{1}{p}-\frac{1}{s}+1}\right).$$

<u>Corollary</u> (2.5.3). If $k(t) = \psi(t)\, t^{-\frac{1}{p}}$, then

$$\left|L(x;\ \xi;\ s) - f(x)\right| = O\left(\xi^{1-\frac{1}{s}-\frac{1}{p}} \psi(\xi)\right).$$

CHAPTER III

ON THE DEGREE OF APPROXIMATION OF SINGULAR INTEGRALS PICARD, POISSON-CAUCHY AND OSTROWSKI'S OPERATOR FOR THE FUNCTIONS, BELONGING TO THE CLASS k(t).

3.1. Let $f(x)$ be a positive Lebesgue integrable function with period 2π and let

$$(3.1.1) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$\equiv \sum_{k=0}^{\infty} A_k(x)$$

be its Fourier series.

In the previous chapter, we have determined the degree of approximation of functions belonging to the class $k(t)$ for the singular integrals (named Gauss-Weierstrass and Generalized Gauss-Weierstrass).

3.2. Butzer, P.L. [6] obtained the degree of approximation for singular integrals Abel-Poisson and Poisson-Cauchy for the functions belonging to the class Lip α.

His theorems state as follows.

Theorem (3.2.1). If $f(x) \in L_p(-\pi, +\pi)$, $1 \leq p < +\infty$ and also belongs to $c[-\pi, +\pi]$ and $\| \sigma(r, x) - f(x) \| = O(1 - r)$ then $f(x)$ is constant, where $\sigma(r, x)$ is Abel-Poisson singular integral.

Theorem (3.2.2). Let $f(x) \in L_p(-\infty, +\infty)$, $1 < p < +\infty$

(a) If $\lim_{\xi \to 0} \frac{1}{\xi} \| f(x, \xi) - f(x) \| = 0$, then $f(x) = 0$.

(b) If $\tilde{f}'(x) \in L_p(-\infty, +\infty)$, then

$$\| f(x; \xi) - f(x) \| \leq \xi \, \| \tilde{f}'(x) \|$$

(c) If $\| f(x; \xi) - f(x) \| = O(\xi)$, then

$$\tilde{f}'(x) \in L_p(-\infty, +\infty).$$

where $f(x; \xi)$ is Poisson-Cauchy singular integral.

3.3. In this Chapter, we have tried the same problem as in the previous chapter for singular integrals Picard, Poisson-Cauchy and Ostrowski's operator and succeeded in obtaining a better approximation generalising the result of Butzer, P.L. [6].

Our results are as follows .

Theorem (3.3.1). If $f(x) \in k(t)$, then

≠ (i) $|P(x;\xi) - f(x)| = O(k(\xi))$

≠ (ii) $|P^{*}(x;\xi) - f(x)| = O(k(\xi))$

+ (iii) $|A(x;\xi;s) - f(x)| = O(k(\xi))$

where $P(x;\xi)$, $P^{*}(x,\xi)$ and $A(x;\xi)$ are Picard, Poisson-Cauchy and Ostrowski's operator respectively.

Proof of part(i) of theorem (3.3.1)

We have

$$P(x;\xi) - f(x) = \frac{1}{2\xi}\int_{-\infty}^{\infty}\left[f(x+t)+f(x-t)-2f(x)\right] .$$

$$e^{-|t|/\xi}\, dt + R(x;\xi) .$$

≠ Khan, Arman [13]
+ Khan, Arman, and S.Umar [18]

$$= \frac{1}{\zeta} \int_0^\infty \phi(t)\, e^{-|t|/\zeta}\, dt + o(\zeta),$$

where $o(\zeta) \to 0$ as $\zeta \to 0$.

$$= \frac{1}{\zeta} \left[\int_0^\zeta + \int_\zeta^\infty \right] \phi(t)\, e^{-|t|/\zeta}\, dt.$$

$$= I_1 + I_2, \text{ say}.$$

evaluating I_1, we have

$$I_1 = \frac{1}{\zeta} \int_0^\zeta \phi(t)\, e^{-|t|/\zeta}\, dt.$$

$$|I_1| \le \frac{1}{\zeta} \int_0^\zeta |\phi(t)|\, e^{-|t|/\zeta}\, dt.$$

$$= \frac{1}{\zeta} \int_0^\zeta k(t)\, e^{-|t|/\zeta}\, dt.$$

$$= \frac{1}{\zeta} k(\zeta) \int_0^\zeta e^{-|t|/\zeta}\, dt.$$

$$= O(k(\zeta)) \int_0^1 e^{-u}\, du.$$

$$= O(k(\zeta)) \cdot O(1).$$

$$= O(k(\zeta)).$$

secondly,

$$I_2 = \frac{1}{\xi}\int_{\xi}^{\infty} \phi(t)\, e^{-|t|/\xi}\, dt$$

$$|I_2| \leq \frac{1}{\xi}\int_{\xi}^{\infty} |\phi(t)|\, e^{-|t|/\xi}\, dt$$

$$= \frac{1}{\xi}\int_{\xi}^{\infty} k(t)\, e^{-|t|/\xi}\, dt.$$

$$= \frac{1}{\xi}\int_{\xi}^{\infty} \frac{k(t)}{t/\xi} \,.\, t/\xi \,.\, e^{-|t|/\xi}\, dt.$$

$$= \frac{1}{\xi}\int_{\xi}^{\infty} \frac{k(t)}{t}.\ \xi.\frac{t}{\xi}\, e^{-|t|/\xi}\, dt:$$

$$= \frac{1}{\xi}\, k(\xi) \int_{\xi}^{\infty} \frac{t}{\xi}\, e^{-|t|/\xi}\, dt.$$

$$= O(k(\xi)) \int_{1}^{\infty} u\, e^{-u}\, du.$$

Therefore,

$$I_2 = O(k(\xi)).\ O(1)$$

$$= O(k(\xi)).$$

Adding the bounds for I_1 and I_2, we have

$$|F(x;\xi) - f(x)| = O(k(\xi)),\quad \xi \to 0.$$

which completes the proof of part (i) of theorem (3.3.1).

Proof of part (ii).

We know that

$$P^{*}(x;\ \xi)-f(x)=\frac{\xi}{\pi}\int_{-\pi}^{\pi}\left[f(x+t)+f(x-t)-2f(x)\right]\frac{dt}{t^{2}+\xi^{2}}+R(x;\ \xi)$$

$$=\frac{2\xi}{\pi}\int_{0}^{\pi}\phi(t)\ \frac{dt}{t^{2}+\xi^{2}}+o(\xi),$$

where $o(\xi)\to 0$ as $\xi\to 0$.

$$=\frac{2\xi}{\pi}\left[\int_{0}^{\xi}+\int_{\xi}^{\pi}\right]\phi(t)\ \frac{dt}{t^{2}+\xi^{2}}$$

$$=I_{1}+I_{2},\ \text{ say}.$$

Now,

$$|I_{1}|\ \leq\ \frac{2\xi}{\pi}\int_{0}^{\xi}|\phi(t)|\ \frac{dt}{t^{2}+\xi^{2}}$$

$$=\frac{2\xi}{\pi}\int_{0}^{\xi}k(t)\ \frac{dt}{t^{2}+\xi^{2}}$$

$$=\frac{2\xi}{\pi}\ k(\xi)\int_{0}^{\xi}\frac{dt}{t^{2}+\xi^{2}}$$

$$= \frac{2\zeta}{\pi} k(\zeta) \left[\frac{1}{\zeta} \tan^{-1} \frac{t}{\zeta} \right]_0^{\zeta}$$

$$= O(k(\zeta)).$$

Again,

$$I_2 = \frac{2\zeta}{\pi} \int_{\zeta}^{\pi} \phi(t) \frac{dt}{t^2+\zeta^2} .$$

$$|I_2| \leq \frac{2\zeta}{\pi} \int_{\zeta}^{\pi} |\phi(t)| \frac{dt}{t^2+\zeta^2} .$$

$$= \frac{2\zeta}{\pi} \int_{\zeta}^{\pi} k(t) \frac{dt}{t^2+\zeta^2} .$$

$$= \frac{2\zeta}{\pi} \frac{k(\zeta)}{\zeta} \int_{\zeta}^{\pi} \frac{t\, dt}{t^2+\zeta^2} .$$

$$= \frac{2\, k(\zeta)}{\pi} \int_{2\zeta^2}^{\pi^2+\zeta^2} \frac{1}{2} \frac{du}{u},$$

$$= \frac{k(\zeta)}{\pi} \left[\log (\pi^2 + \zeta^2) - \log 2\zeta^2 \right] .$$

$$= O(k(\zeta))$$

Adding the bounds, we see that

$$I_1 + I_2 = O(k(\zeta)).$$

Hence,

$$|F^{*}(x\,;\,\xi) - f(x)| = O(\,k(\xi)\,)$$

which completes the proof of part (ii).

Proof of part (iii) of theorem (3.3.1)

In order to prove this part, we will prove the following Lemmas.

Lemma (A). $\dfrac{1}{\Gamma\xi}\displaystyle\int_0^1 x^{\xi-1}\, e^{-x}\, dx = 1, \qquad \text{as } \xi \to 0.$

Lemma (B). $\dfrac{1}{\Gamma\xi}\displaystyle\int_1^{\infty} x^{2\xi-1}\, e^{-x}\, dx = 0.$

Proof of Lemma (A).

Using (1.7.6), we have

$$P(x\,;\,\xi) = \frac{1}{\Gamma\xi}\int_0^1 x^{\xi-1}\, e^{-x}\, dx$$

Also, using (1.7.7), we get

$$\gamma^{*}(\xi;\,1) = P(\xi,'\,1)$$

Therefore,

$$P(\xi,'\,1) = 1$$

Hence,

$$P(\zeta, 1) = \frac{1}{\Gamma\zeta} \int_0^1 x^{\zeta-1} e^{-x}\, dx$$

$$= 1$$

which completes the proof of Lemma (A).

<u>Proof of Lemma (B).</u>

Using (1.7.3), we have

$$\Gamma(2\zeta, 1) = \Gamma 2\zeta - \gamma(2\zeta, 1)$$

$$= \int_1^\infty x^{2\zeta-1} e^{-x}\, dx$$

Also, using (1.7.4), we notice that,

$$\gamma(2\zeta, 1) = \int_0^1 x^{2\zeta-1} e^{-x}\, dx$$

Again, using (1.7.5),

$$\gamma^* (2\zeta, 1) = \frac{\gamma(2\zeta, 1)}{\Gamma 2\zeta}$$

Therefore,

$$\frac{1}{\Gamma\zeta}\int_1^\infty x^{2\zeta-1} e^{-x}\, dx = \frac{1}{\Gamma\zeta}\, \Gamma(2\zeta, 1)$$

$$= \frac{\sqrt{2\xi} - \gamma(2\xi;\ 1)}{\sqrt{\xi}}$$

$$= \frac{\sqrt{2\xi}}{\sqrt{\xi}} - \frac{\sqrt{2\xi}}{\sqrt{\xi}} \cdot \frac{\gamma(2\xi,\ 1)}{\sqrt{2\xi}}$$

$$= \frac{\sqrt{2\xi}}{\sqrt{\xi}}\left[1 - \frac{\gamma(2\xi,\ 1)}{\sqrt{2\xi}}\right]$$

$$= \frac{1}{2}\left[1 - \gamma^{*}(2\xi,\ 1)\right] \quad , \quad \text{by } (1.7.8)$$

$$= 0 \qquad \text{as } \gamma^{*}(2\xi,\ 1) = 1$$

which completes the proof of Lemma (B).

<u>Proof of part(iii) of theorem (3.1.1)</u>

We know that,

$$A(x\ ;\ \xi) - f(x) = \frac{1}{2\xi^{\xi}\sqrt{\xi}} \int_{-\infty}^{\infty}\Big[f(x+t) + f(x-t) - 2f(x)\Big]\, e^{-(|t|/\xi)^{1/\xi}}\, dt + R(x;\ \xi).$$

$$= \frac{1}{\xi^{\xi}\sqrt{\xi}} \int_{0}^{\infty} \phi(t)\, e^{-(|t|/\xi)^{1/\xi}}\, dt + o(\xi).$$

where $o(\zeta) \to 0$ as $\zeta \to 0$

$$= \frac{1}{\zeta^2 \Gamma\zeta} \int_0^\infty \phi(t)\, e^{-(|t|/\zeta)^{1/\zeta}}\, dt.$$

$$= \frac{1}{\zeta^2 \Gamma\zeta}\left[\int_0^\zeta + \int_\zeta^\infty\right]\phi(t)\, e^{-(|t|/\zeta)^{1/\zeta}}\, dt$$

$$= I_1 + I_2, \text{ say}$$

Now, evaluating I_1, we have,

$$I_1 = \frac{1}{\zeta^2 \Gamma\zeta} \int_0^\zeta \phi(t)\, e^{-(|t|/\zeta)^{1/\zeta}}\, dt$$

$$|I_1| \leq \frac{1}{\zeta^2 \Gamma\zeta} \int_0^\zeta |\phi(t)|\, e^{-(|t|/\zeta)^{1/\zeta}}\, dt.$$

$$= \frac{1}{\zeta^2 \Gamma\zeta} \int_0^\zeta k(t)\, e^{-(|t|/\zeta)^{1/\zeta}}\, dt.$$

$$= \frac{1}{\zeta^2 \Gamma\zeta}\, k(\zeta) \int_0^\zeta e^{-(|t|/\zeta)^{1/\zeta}}\, dt.$$

$$= \frac{1}{\Gamma\zeta}\, k(\zeta) \int_0^1 x^{\zeta-1}\, e^{-x}\, dx,$$

on putting $(t/\zeta)^{1/\zeta} = x$,

$$= k(\xi) \frac{1}{\Gamma\xi} \int_0^1 x^{\xi-1} e^{-x} dx$$

$$= O(k(\xi)) \cdot O(1) \quad , \text{ by Lemma (A)}$$

$$= O(k(\xi))$$

Therefore,

$$I_1 = O(k(\xi))$$

Secondly,

$$I_2 = \frac{1}{\xi^2 \Gamma\xi} \int_\xi^\infty \phi(t)\, e^{-(|t|/\xi)^{1/\xi}} dt.$$

$$|I_2| \leq \frac{1}{\xi^2 \Gamma\xi} \int_\xi^\infty |\phi(t)|\, e^{-(|t|/\xi)^{1/\xi}} dt.$$

$$\doteq \frac{1}{\xi^2 \Gamma\xi} \int_\xi^\infty k(t)\, e^{-(|t|/\xi)^{1/\xi}} dt.$$

$$= \frac{1}{\xi^2 \Gamma\xi} \frac{k(\xi)}{\xi} \int_\xi^\infty t\, e^{-(|t|/\xi)^{1/\xi}} dt.$$

$$= \frac{1}{\xi^2 \Gamma\xi} \frac{k(\xi)}{\xi} \xi^3 \int_1^\infty x^{2\xi-1} e^{-x} dx$$

$$= 0, \quad \text{by Lemma (B)}$$

therefore,

$$I_2 = 0$$

Adding the bounds, we have

$$I_1 + I_2 = O(\,k(\xi)\,)$$

Hence,

$$|A\,(x\,;\,\xi) - f(x)| = O(\,k(\xi)\,)$$

which completes the proof of part (iii) of theorem (3.1.1).

3.4. It may be remarked that on giving different values to $k(t)$ we have the following corollaries for our theorem (3.1.1).

<u>Corollary (3.4.1)</u>. If $k(t) = t^{\alpha}$, then for each part of the theorem (3.1.1) we have $O(\xi^{\alpha})$.

<u>Corollary (3.4.2)</u>. If $k(t) = t^{\alpha - \frac{1}{p}}$, then we have $O(\xi^{\alpha - \frac{1}{p}})$

<u>Corollary (3.4.3)</u>. If $k(t) = \psi(t)\,t^{-1/p}$, we have $O(\xi^{-1/p}\,\psi(\xi)\,)$.

CHAPTER IV

ON THE DEGREE OF APPROXIMATION TO A FUNCTION BELONGING TO THE CLASSES h(t) AND g(t) USING GAUSS-WEIERSTRASS SINGULAR INTEGRAL.

4.1 : In the foregoing chapter, the results obtained by Butzer, P.L [5], [6] for De La Vallée - Poussin and Gauss-weierstrass singular integrals respectively have been extended to determine the degree of approximation by Gauss-weierstrass singular integral of functions belonging to the classes $h(t)$ and $g(t)$, where these classes are positive increasing functions with the following conditions.

For the class $h(t)$, we have

(a) $h(\alpha x) = h(\alpha)\, h(x)$, where α is constant.

(b) $\int_0^u [f(x+t) + f(x-t) - 2f(x)]\,dt = O(h(u))$ as $u \to 0$.

Also, for the class $g(t)$, we have,

(a) $g(\alpha x) = g(\alpha)\, g(x)$, where α is constant.

(b) $\int_0^u \lambda(t)\,dt = O(g(u))$, satisfying

$$\lambda(t) = \int_{-\infty}^{\infty} |f(x+t) + f(x-t) - 2f(x)|\,dx$$

It may be remarked that,

(i) If $\frac{h(t)}{t^2} = \frac{g(t)}{t^2} = t^{\alpha}$, $0 < \alpha < 1$, then our classes reduce to Lipα.

(ii) If $\frac{h(t)}{t^2} = \frac{g(t)}{t^2} = t^{\alpha-1/p}$, and if $f(x) \in L^p (p>1)$, our classes reduce to Lip$^{+}(\alpha, p)$.

(iii) If $\frac{h(t)}{t^2} = \frac{g(t)}{t^2} = \psi(t)\, t^{-1/p}$, where $\psi(t)$ is a positive increasing function and $f(x) \in L^p (p > 1)$, then our classes reduce to Lip$^{\neq}$ $(\psi(t), p)$.

4.2. Butzer, P.L [5] proceeded to discuss the problem of approximation for functions $f(x) \in L_p$ having an integral modulus of continuity or belonging to the class Lip $(\alpha; p)$ for $0 < \alpha \leq 1$, $p \geq 1$.

He studied the approximation in the mean of $f(x)$ by De La Vallée-Poussin singular integral.

If $f(x)$ is periodic (period 2π) and $f \in L_p$, $p \geq 1$ with

$$\| f \|_p = [\int_{-\pi}^{\pi} |f(x)|^p dx]^{1/p}$$

+ Hardy and Little wood [9]

$\neq$ Khan, Huzoor H. [21].

the integral modulus of continuity of $f(x)$ is defined as,

$$w_p(\delta) = w_p(\delta; f) = \sup_{0<|h|\leq\delta} \| f(x+h) - f(x) \|_p \,.$$

and generalised integral modulus of continuity of $f(x)$ as

$$w_p^*(\delta) = w_p(\delta; f) = \sup_{0<|h|\leq\delta} \| f(x+h) + f(x-h) - 2f(x)\|_p \,.$$

It is clear that $w_p^*(\delta) \leq 2\, w_p(\delta)$. If $w_p(\delta) \leq M\delta^{\alpha}$, $0 < \alpha \leq 1$, then $f(x)$ is said to satisfy an integral Lipchits condition or $f \in \mathrm{Lip}\,(\alpha\,;\,p)$. Khan, Huzoor H [22] generalised the integral modulus of continuity class as

$$w^*(\delta, p, \beta) = \sup_{0<|h|\leq\delta} \| [f(x+h) - f(x)] \sin^{\beta} t \|$$

for $\beta \geq 0$. It can be reduced that for $\beta = 0$, the class defined above reduces to $w_p^*(\delta)$.

It has been shown by Quade [34] that the class of functions satisfying the condition $\mathrm{Lip}(\alpha, p)$, $0 < \alpha < 1$ is identical with the class of functions approximable in the mean pth power with error $O(\frac{1}{n^{\alpha}})$ by trignometric polynomials of order n. For the class $w^*(\delta, p, \beta)$ the above remark of Quade [34] is also true.

For $\alpha = 1$, Zygmund [40] has shown that this class is equivalent to the class of functions approximable in the mean pth power with error $O(\frac{1}{n})$ by trignometric polynomials of order n.

In fact, his theorem states as follows:

THEOREM (4.2.1) : If $f(x)$ has generalised modulus of continuity $w_p^*(\delta)$, $p \geq 1$, then

$$\| V_n(x) - f \|_p = O(w_p^*(\frac{1}{\sqrt{n}})).$$

where $V_n(x)$ is a De La Vallée-Poussin singular integral.

4.3 : Further, Butzer, P.L [6] extended theorem (4.2.1) under certain conditions for Gauss-weierstrass singular integral.

In this regard, he proved the following theorem:

Theorem (A) : If $f(x) \in L_p(-\infty, +\infty)$, $1 < p < \infty$ and

$$\| W(x\,;\,\xi) - f(x) \| = O(\xi),\ \text{then}$$

$f(x)$, $f'(x)$ and $f''(x)$ belong to $L_p(-\infty, +\infty)$.

In this chapter, we have determined the degree of approximation of functions belonging to the class $h(t)$ and $g(t)$ by Gauss-Weierstrass singular integrals generalising The results of Butzer, P.L [5], [6].

Our theorems are as follows:

*<u>Theorem</u> (4.3.1): If $f(x)$ belongs to the class $h(t)$, then

$$|W(x;\xi) - f(x)| = O\left(\frac{h(\sqrt{\xi})}{\sqrt{\xi}}\right), \quad \xi \to 0+,$$

provided $h(t)$ satisfies the following conditions:

$$\frac{h(t_0)}{h(\sqrt{\xi})} \quad e^{-\left(\frac{t_0}{\sqrt{\xi}}\right)^2} = o(1)$$

$$\frac{d}{dt}(h(t)) = O\left(\frac{h(t)}{t}\right), \text{ and}$$

$$\int_{t_0/\sqrt{\xi}}^{\infty} \frac{h(v)}{v} \, e^{-v^2} \, dv < +\infty$$

where, we break the interval $(0, \infty)$ into $(0, t_0)$ and (t_0, ∞)

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* Khan, Arman and S. Umar [16].

*Theorem (4.3.2): If $f(x)$ belongs to the class $\mathcal{G}(t)$, then

$$\int_{-\infty}^{\infty} |W(x;\xi) - f(x)|\, dx = O\left(\frac{g(\sqrt{\xi})}{\sqrt{\xi}}\right), \quad \xi \to 0+.$$

provided $g(t)$ satisfies the following conditions:

$$\frac{g(t_0)}{g(\sqrt{\xi})} \; e^{-(t_0/\sqrt{\xi})^2} = o(1)$$

$$\frac{d}{dt}(g(t)) = O\left(\frac{g(t)}{t}\right), \text{ and}$$

$$\int_{t_0/\sqrt{\xi}}^{\infty} \frac{g(v)}{v} \; e^{-v^2}\, dv < +\infty .$$

Proof of theorem (4.3.1):

We know that,

$$W(x;\xi) - f(x) = \frac{2}{\sqrt{\pi\xi}} \int_0^{\infty} \phi_x(t)\, e^{-t^2/\xi} \, dt.$$

Multiplying both sides by $\dfrac{\sqrt{\xi}}{h(\sqrt{\xi})}$, we have,

$$\frac{\sqrt{\xi}}{h(\sqrt{\xi})}\left[W(x;\xi) - f(x)\right] = \frac{2}{\sqrt{\pi}\, h(\sqrt{\xi})} \int_0^{t_0} e^{-t^2/\xi} \, d\Phi(t) +$$

$$\frac{1}{\sqrt{\pi}\, h(\sqrt{\xi})}\left[\int_{t_0}^{\infty} f(x+t)\, e^{-t^2/\xi} \, dt + \int_{t_0}^{\infty} f(x-t)\, e^{-t^2/\xi} \, dt -\right.$$

* Khan, Arman and S. Umar [16].

$$\int_{t_0}^{\infty} 2f(x)\ e^{-t^2/\xi}\ dt.]$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say.}$$

For evaluating I_1, we have

$$I_1 = \frac{2}{\sqrt{\pi}\ h(\sqrt{\xi}\ )} [e^{-t_0^2/\xi}\ \bar{\Phi}(t_0) - \int_0^{t_0} \Phi(t)\ d_t(\ e^{-t^2/\xi}\ )]$$

$$= I_{11} + I_{12}, \text{ say.}$$

Now, evaluating I_{11}, we get

$$I_{11} = \frac{2}{\sqrt{\pi}} \quad \frac{h(t_0)}{h(\sqrt{\xi}\ )}\ e^{-t_0^2/\xi} \qquad \frac{\bar{\Phi}(t_0)}{h(t_0)} \ .$$

$$= o(1).$$

Next,

$$I_{12} = \frac{2}{\sqrt{\pi}\ h(\sqrt{\xi}\ )}\ O[\int_0^{t_0} h(t)\ d_t(\ e^{-t^2/\xi}\ )]$$

$$= O[\frac{h(t_0)}{\sqrt{\pi}\ h(\sqrt{\xi}\ )}\ \ e^{-(t_0/\sqrt{\xi}\ )^2}] + O[\int_0^{t_0} (\ \frac{\sqrt{\xi}}{t})\ \frac{h(t)}{h(\sqrt{\xi}\ )}$$

$$e^{-(t/\sqrt{\xi}\ )^2}\ d\ (\ \frac{t}{\sqrt{\xi}}\ )] \ .$$

$$= o\ (1) + O\ (1)$$

$$= O\ (1), \quad \xi \to o$$

Adding the bounds for I_{11} and I_{12}, we have

$$I_1 = I_{11} + I_{12}$$

$$= o\ (1) + O\ (1)$$

$$= O\ (1)$$

Now,

$$|\ I_2\ | \leqq \frac{1}{\sqrt{\pi}\ h(\sqrt{\xi}\ )}\ e^{-t_o^2/\xi} \int_{t_o}^{\infty} |\ f(x+t)|\ dt.$$

$$\leqq \frac{M}{\sqrt{\pi}\ h(t_o)}\ \frac{h(t_o)}{h(\sqrt{\xi}\ )}\ e^{-(t_o/\sqrt{\xi}\ )^2}$$

$$= o\ (1)$$

Also,

$$I_3 \leqq \frac{1}{\sqrt{\pi}\ h(\sqrt{\xi}\ )}\ e^{-t_o^2/\xi} \int_{t_o}^{\infty} |\ f(x-t)|\ dt.$$

$$= o\ (1)$$

Again,

$$|\ I_4| \leqq \frac{2|f(x)|}{\sqrt{\pi}\ h(\sqrt{\xi}\ )} \int_{t_o}^{\infty} e^{-t^2/\xi}\ dt.$$

$$\leq \frac{2|f(x)|}{\sqrt{\pi}\, h(t_0)} \int_{t_0}^{\infty} \frac{h(t)}{t\, h(\sqrt{\xi}\,)} \quad e^{-t^2/\xi} \quad dt.$$

$$= \frac{2|f(x)|t_0}{\sqrt{\pi}\, h(t_0)} \int_{t_0/\sqrt{\xi}}^{\infty} \frac{h(v)}{v} \quad e^{-v^2} \ dv.$$

$$= o\ (1)$$

Adding the bounds, we get

$$I_1+I_2+I_3+I_4 = O\ (1) + o(1) + o(1) + o(1)$$

$$= O\ (1)$$

Hence,

$$|W(x;\xi\,) - f(x)| = O\Big(\frac{h(\sqrt{\xi}\,)}{\sqrt{\xi}} \Big), \quad \xi \to o + .$$

which completes the proof of theorem (4.3.1).

<u>Proof of Theorem</u> (4.3.2).

we have,

$$\frac{\sqrt{\xi}}{g(\sqrt{\xi}\,)} \int_{-\infty}^{\infty} |W(x;\xi\,)-f(x)|dx \leq \frac{2}{\sqrt{\pi} g(\sqrt{\xi}\,)} \int_{-\infty}^{\infty} dx[\int_{0}^{\infty}|\phi_x(t)|\ e^{-t^2/\xi}\, dt].$$

$$= \frac{2}{\sqrt{\pi} g(\sqrt{\xi}\,)} \int_{0}^{\infty} e^{-t^2/\xi} \ dt \int_{-\infty}^{\infty} |\phi_x(t)|\ dx.$$

$$= \frac{2}{\sqrt{\pi} g(\sqrt{\xi}\,)} \int_{0}^{\infty} e^{-t^2/\xi} \ \lambda(t)\ dt.$$

we can write, as in the proof of theorem (4.3.1)

$$I_1 = O(1)$$

and,

$$I_2 + I_3 + I_4 = \frac{2}{\sqrt{\pi}\, g(\sqrt{\xi}\,)} \int_{t_0}^{\infty} \lambda(t)\; e^{-t^2/\xi}\; dt.$$

$$\leq \frac{2\, M\, t_0}{\sqrt{\pi}\, g(t_0)} \int_{t_0}^{\infty} \frac{g(t)}{t\, g(\sqrt{\xi}\,)}\; e^{-t^2/\xi}\; dt.$$

$$= O\,(1), \quad \xi \to 0$$

Therefore,

$$I_1 + I_2 + I_3 + I_4 = O\,(1) + O\,(1)$$

$$= O\,(1)$$

Hence,

$$\int_{-\infty}^{\infty} |W(x;\xi\,) - f(x)|\; dx = O\,\left(\frac{g(\sqrt{\xi}\,)}{\sqrt{\xi}}\right), \quad \xi \to 0+$$

which completes the proof of the theorem (4.3.2).

4.4. In this section of the chapter, we have used the conditions $O\left(\frac{h(t)}{t}\right)$ and $O\left(\frac{g(t)}{t}\right)$ on $\phi_x(t)$ in place of $O(h(t))$ and $O(g(t))$ respectively.

In fact, our theorems are as follows:

*<u>Theorem</u> (4.4.1) : If $f(x)$ belongs to the class $h(t)$, then

$$|W(x;\xi) - f(x)| = O\left(\frac{h(\sqrt{\xi})}{\xi}\right), \quad \xi \to 0+$$

provided that,

$$\left(\frac{\sqrt{\xi}}{t_0}\right) \frac{h(t_0)}{h(\sqrt{\xi})} e^{-(t_0/\sqrt{\xi})^2} = o(1)$$

$$\frac{d}{dt}\left(\frac{h(t)}{t}\right) = O\left(\frac{h(t)}{t^2}\right), \text{ and}$$

$$\int_{t_0/\sqrt{\xi}}^{\infty} \frac{h(v)}{v^2} e^{-v^2} dv < +\infty$$

*<u>Theorem</u> (4.4.2) : If $f(x)$ belongs to the class $g(t)$ then,

$$\int_{-\infty}^{\infty} |W(x;\xi) - f(x)| \, dx = O\left(\frac{g(\sqrt{\xi})}{\xi}\right)$$

under the conditions,

$$\left(\frac{\sqrt{\xi}}{t_0}\right) \frac{g(t_0)}{g(\sqrt{\xi})} e^{-(t_0/\sqrt{\xi})^2} = o(1).$$

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* Khan, Arman and S. Umar [16].

$$\frac{d}{dt}\left(\frac{g(t)}{t}\right) = O\left(\frac{g(t)}{t^2}\right), \text{ and}$$

$$\int_{t_0/\sqrt{\xi}}^{\infty} \frac{g(v)}{v^2} e^{-v^2} dv < +\infty .$$

Proof of theorem (4.4.1) :

We know that,

$$W(x;\xi) - f(x) = \frac{2}{\sqrt{\pi\xi}} \int_0^{\infty} \phi_x(t) \; e^{-t^2/\xi} \; dt.$$

on multiplying both sides by $\frac{\xi}{h(\sqrt{\xi})}$, we have,

$$\frac{\xi}{h(\sqrt{\xi})}\left[W(x;\xi) - f(x)\right] = \frac{2\sqrt{\xi}}{\sqrt{\pi}\, h(\sqrt{\xi})} \int_0^{t_0} e^{-t^2/\xi} \, d\Phi(t) +$$

$$\frac{\sqrt{\xi}}{\sqrt{\pi}\, h(\sqrt{\xi})}\left[\int_{t_0}^{\infty} f(x+t)\, e^{-t^2/\xi} \, dt + \int_{t_0}^{\infty} f(x-t)\, e^{-t^2/\xi} \, dt - \int_{t_0}^{\infty} 2f(x) e^{-t^2/\xi} dt\right]$$

$$= I_1 + I_2 + I_3 + I_4 , \text{ say.}$$

integrating I_1 by parts, we find,

$$I_1 = \frac{2\sqrt{\xi}}{\sqrt{\pi}\, h(\sqrt{\xi}\,)} \left[e^{-t_0^2/\xi}\ \Phi(t_0) - \int_0^{t_0} \Phi(t)\, d_t\, (e^{-t^2/\xi}\ ) \right]$$

$$= I_1' + I_1'' , \quad \text{say}$$

Now,

$$I_1' = \frac{2}{\sqrt{\pi}} \left(\frac{\sqrt{\xi}}{t_0} \right) \frac{h(t_0)}{h(\sqrt{\xi}\ )}\ e^{-t^2/\xi} \quad \frac{\Phi(t_0)\ t_0}{h(t_0)}$$

$$= o(1)$$

Again,

$$I_1 = \frac{2\sqrt{\xi}}{\sqrt{\pi}\, h(\sqrt{\xi}\,)}\ O\left[\int_0^{t_0} \frac{h(t)}{t}\ d_t\ (e^{-t^2/\xi}\ ) \right]$$

$$= O\left[\left(\frac{\sqrt{\xi}}{t_0} \right) \frac{h(t_0)}{h(\sqrt{\xi})}\ e^{-(t_0/\sqrt{\xi}\ )^2} \right] +$$

$$O\left[\int_0^{t_0} \left(\frac{\sqrt{\xi}}{t} \right)^2 \frac{h(t)}{h(\sqrt{\xi})}\ e^{-(t/\sqrt{\xi}\ )^2}\ d\left(\frac{t}{\sqrt{\xi}} \right) \right].$$

$$= o(1) + O(1)$$

$$= O\ (1)$$

Therefore,

$$I_1 = I_1' + I_1''$$

$$= o(1) + O(1)$$

$$= O\ (1)$$

Now,

$$|I_2| \leq \frac{\sqrt{\xi}}{\sqrt{\pi}\ h(\sqrt{\xi}\ )} \quad e^{-t_o^2/\xi} \quad \int_{t_o}^{\infty} |f(x+t)|\ dt.$$

$$\leq \frac{M\ t_o}{\sqrt{\pi}\ h(t_o)} \quad \left(\frac{\sqrt{\xi}}{t_o} \right)^2 \quad \frac{h(t_o)}{h(\sqrt{\xi}\ )} \quad e^{-(t_o/\sqrt{\xi}\ )^2}$$

$$= o\ (1)$$

Similarly,

$$I_3 = o\ (1)$$

Again,

$$|\ I_4\ | \leq \frac{2|f(x)|\sqrt{\xi}}{\sqrt{\pi}\ h(\sqrt{\xi}\ )} \quad \int_{t_o}^{\infty} \quad e^{-t^2/\xi} \quad dt.$$

$$\leq \frac{2|f(x)|\ t_o^2}{\sqrt{\pi}\ h(t_o)} \quad \int_{t_o}^{\infty} \left(\frac{\sqrt{\xi}}{t^2} \right) \frac{h(t)}{h(\sqrt{\xi}\ )} \quad e^{-t^2/\xi}\ \ dt.$$

$$= \frac{2|f(x)|\ t_o^2}{\sqrt{\pi}\ h(t_o)} \quad \int_{t_o/\sqrt{\xi}}^{\infty} \frac{h(v)}{v^2}\ e^{-v^2}\ \ dv.$$

$$= o\ (1).$$

Adding the bounds, we have

$$I_1 + I_2 + I_3 + I_4 = O\ (1)$$

Hence,

$$|W(x;\xi) - f(x)| = O\left(\frac{h(\sqrt{\xi})}{\xi}\right)$$

which completes the proof of the theorem (4.4.1)

<u>Proof of theorem</u> (4.4.2):

We have

$$\frac{\xi}{g(\sqrt{\xi}\ )}\int_{-\infty}^{\infty}|W(x;\xi)-f(x)|dx \leq \frac{2\sqrt{\xi}}{\sqrt{\pi}g(\sqrt{\xi})}\int_{-\infty}^{\infty}dx\int_{0}^{\infty}|\phi_x(t)|\ e^{-t^2/\xi}\,dt.$$

$$= \frac{2\sqrt{\xi}}{\sqrt{\pi}\ g(\sqrt{\xi})}\int_{0}^{\infty}e^{-t^2/\xi}\,dt\int_{-\infty}^{\infty}|\phi_x(t)|\,dx.$$

$$= \frac{2\sqrt{\xi}}{\sqrt{\pi}\ g(\sqrt{\xi})}\int_{0}^{\infty}e^{-t^2/\xi}\,\lambda(t)\,dt.$$

Therefore,

$I_1 = O\ (1)$, as in the proof of theorem (4.4.1).

and also,

$$I_2 + I_3 + I_4 = \frac{2\sqrt{\xi}}{\sqrt{\pi}\, g(\sqrt{\xi})} \int_{t_0}^{\infty} \lambda(t)\, e^{-t^2/\xi}\, dt.$$

$$\leq \frac{2M\, t_0^2}{\sqrt{\pi}\, g(t_0)} \int_{t_0}^{\infty} \left(\frac{\sqrt{\xi}}{t^2}\right) \frac{g(t)}{g(\sqrt{\xi})}\, e^{-t^2/\xi}\, dt.$$

$$= O(1)$$

Hence, adding the bounds, we have,

$$\int_{-\infty}^{\infty} |W(x; \xi) - f(x)|\, dx = O\left(\frac{g(\sqrt{\xi})}{\xi}\right),$$

which completes the proof of the theorem (4.4.2).

<u>4.5</u>. The following are the corollaries of our theorem (4.3.1) and (4.4.1) for the conditions $O(h(t))$ and $O\left(\frac{h(t)}{t}\right)$ on $\phi_x(t)$ respectively.

<u>Corollary</u> (4.5.1) : If $\frac{h(t)}{t^2} = t^{\alpha}$, then

(a) $|W(x;\xi) - f(x)| = O(\xi^{1+\alpha/2})$

(b) $|W(x;\xi) - f(x)| = O(\xi^{\alpha/2})$

Corollary (4.5.2) : If $\dfrac{h(t)}{t^2} = t^{\alpha - 1/p}$ we have,

(a) $|W(x;\xi) - f(x)| = O(\xi^{\alpha/2 + 1/2 - 1/2p})$

(b) $|W(x;\xi) - f(x)| = O(\xi^{\alpha/2 - 1/2p})$

Corollary (4.5.3) : If $\dfrac{h(t)}{t^2} = \psi(t)\, t^{-1/p}$, we have

(a) $|W(x;\xi) - f(x)| = O(\xi^{\frac{1}{2} - 1/2p}\, \psi(\sqrt{\xi}))$.

(b) $|W(x;\xi) - f(x)| = O(\xi^{-1/2p}\, \psi(\sqrt{\xi}))$.

4.6. Now, considering the conditions, $O(g(u))$ and $O(\frac{g(u)}{u})$ respectively on $\phi_x(t)$, and putting different values of $\frac{g(t)}{t^2}$, we have the following corollaries of our theorems (4.3.2) and (4.4.2) respectively.

Corollary (4.6.1) : If $\dfrac{g(t)}{t^2} = t^{\alpha}$, then

(a) $\int_{-\infty}^{\infty} | W(x; \xi) - f(x) | \, dx = O(\xi^{1+\alpha/2})$

(b) $\int_{-\infty}^{\infty} | W(x; \xi) - f(x) | \, dx = O(\xi^{\alpha/2})$

<u>Corollary</u> (4.6.2) : If $\dfrac{g(t)}{t^2} = t^{\alpha - 1/p}$, we have

(a) $\int_{-\infty}^{\infty} | W(x; \xi) - f(x) | \, dx = O(\xi^{\alpha/2 + 1/2 - 1/2p})$

(b) $\int_{-\infty}^{\infty} | W(x; \xi) - f(x) | \, dx = O(\xi^{\alpha/2 - 1/2p})$

<u>Corollary</u> (4.6.3) : If $\dfrac{g(t)}{t^2} = \psi(t)\, t^{-1/p}$, we get

(a) $\int_{-\infty}^{\infty} | W(x; \xi) - f(x) | \, dx = O(\xi^{1/2 - 1/2p} \psi(\sqrt{\xi}))$.

(b) $\int_{-\infty}^{\infty} | W(x; \xi) - f(x) | \, dx = O(\xi^{-1/2p} \psi(\sqrt{\xi}))$.

CHAPTER V

ON THE DEGREE OF APPROXIMATION TO A FUNCTION BY GENERALIZED GAUSS-WEIERSTRASS SINGULAR INTEGRAL

5.1. In the previous chapter, we have considered the operator Gauss-weierstrass singular integral and determined the degree of approximation of functions (belonging to the classes h(t) and g(t) generalising the results of Butzer, P.L [5], [6].

In this chapter, we have extended our results for the more general operater named Generalised Gauss-weierstrass singular integral for the above said classes h(t) and g(t) and tried to get better results generalising our theorems of chapter IV.

It may be mentioned here that all the theorems (4.3.1), (4.3.2), (4.3.3) and (4.3.4) with their corollaries are reduced for $s = 2$ by our theorems of the foregoing chapter.

5.2. In this section, we have considered the conditions $O(h(t))$ and $O(g(t))$ on $\phi_x(t)$ and determined the degree of approximation.

In fact, we state and prove the following theorems:

*<u>Theorem</u> (5.2.1) : If $f(x) \in h(t)$, then

$$| L(x; \xi; s) - f(x) | = O\left(\frac{h(\xi^{1/s})}{\xi^{1/s}}\right)$$

provided $h(t)$ satisfies the following conditions:

$$\frac{h(t_0)}{h(\xi^{1/s})} \; e^{-|t_0|^s/\xi} = o(1)$$

$$\frac{d}{dt}(h(t)) = O\left(\frac{h(t)}{t}\right), \text{ and}$$

$$\int_{t_0/\xi^{1/s}}^{\infty} \frac{h(v)}{v} \; e^{-v^s} dv < +\infty$$

where, we break the interval (o, ∞) into (o, t_o) and (t_o, ∞).

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* Khan, Arman and S. Umar [19].

*__Theorem__ (5.2.2) : If $f(x) \in g(t)$, then

$$\int_{-\infty}^{\infty} |L(x; \xi; s) - f(x)| \, dx = O\left(\frac{g(\xi^{1/s})}{\xi^{1/s}} \right)$$

under the conditions:

$$\frac{g(t_0)}{g(\xi^{1/s})}, \quad e^{-|t_0|^s/\xi} = o(1)$$

$$\frac{d}{dt}(g(t)) = O\left(\frac{g(t)}{t}\right), \text{ and}$$

$$\int_{t_0/\xi^{1/s}}^{\infty} \frac{g(v)}{v} \, e^{-v^s} dv < +\infty$$

__Proof of theorem__ (5.2.1):

We know that

$$L(x; \xi; s) - f(x) = \frac{s}{\Gamma(1/s) \, \xi^{1/s}} \int_0^{\infty} \phi_x(t) \, e^{-|t|^s/\xi} \, dt.$$

on dividing by $\dfrac{h(\xi^{1/s})}{\xi^{1/s}}$, we have

$$\frac{\xi^{1/s}}{h(\xi^{1/s})}\left[L(x;\xi;s)-f(x)\right] = \frac{s}{\Gamma(1/s)\, h(\xi^{1/s})} \int_0^{t_0} e^{-|t|^s/\xi}\, d\Phi(t) +$$

$$\frac{s}{\Gamma(1/s)\, h(\xi^{1/s})}\left[\int_{t_0}^{\infty} f(x+t)\, e^{-|t|^s/\xi}\, dt + \int_{t_0}^{\infty} f(x-t)\, e^{-|t|^s/\xi}\, dt - \right.$$

$$\left. \int_{t_0}^{\infty} 2\, f(x)\, e^{-|t|^s/\xi}\, dt \right].$$

$= I_1 + I_2 + I_3 + I_4,$ say.

For evaluating I_1, we have

$$I_1 = \frac{s}{\Gamma(1/s)\, h(\xi^{1/s})}\left[e^{-|t_0|^s/\xi}\, \Phi(t_0) - \int_0^{t_0} \Phi(t)\, d_t\left(e^{-|t|^s/\xi} \right) \right]$$

$= I_{11} + I_{12},$ say.

Now,

$$I_{11} = \frac{s}{\Gamma(1/s)}\ \frac{h(t_0)}{h(\xi^{1/s})}\ e^{-\left(|t_0|/\xi^{1/s}\right)^s}\ \frac{\Phi(t_0)}{h(t_0)}$$

$= o(1),\ \xi \to 0+$

Next,

$$I_{12} = \frac{s}{\sqrt{1/s}\; h(\xi^{1/s})}\, O\left[\int_{0}^{t_0} h(t)\, d_t\, \left(e^{-|t|^s/\xi}\right)\right]$$

$$= O\left[\frac{h(t_0)}{h(\xi^{1/s})}\; e^{-\left(|t_0|/\xi^{1/s}\right)^s}\right] +$$

$$O\left[\int_{0}^{t_0}\left(\frac{\xi^{1/s}}{t}\right)\frac{h(t)}{h(\xi^{1/s})}\; e^{-\left(|t|/\xi^{1/s}\right)^s}\, d\left(\frac{t}{\xi^{1/s}}\right)\right]$$

$$= o\,(1) + O\,(1)$$

$$= O\,(1), \quad \xi \to 0+.$$

Adding the bounds for I_{11} and I_{12}, we get

$$I_{11} + I_{12} = o\,(1) + O\,(1)$$

$$= O\,(1)$$

Again

$$|I_2| \leq \frac{s}{2\sqrt{1/s}\; h(\xi^{1/s})}\; e^{-|t_0|^s/\xi} \int_{t_0}^{\infty} |f(x+t)|\, dt.$$

$$\leq \frac{M\, s}{2\sqrt{1/s}\; h(t_0)}\; \frac{h(t_0)}{h(\xi^{1/s})}\; e^{-|t_0|^s/\xi}$$

$$= o\,(1).$$

Similarly, we can write,

$$| I_3 | \leq \frac{s}{2\Gamma 1/s \, h(\xi^{1/s})} \, e^{-|t_0|^s/\xi} \int_{t_0}^{\infty} |f(x-t)| \, dt.$$

$$= o \, (1).$$

Again,

$$| I_4 | \leq \frac{s|f(x)|}{\Gamma 1/s \, h(\xi^{1/s})} \int_{t_0}^{\infty} e^{-|t|^s/\xi} \quad dt.$$

$$\leq \frac{s \, H \, t_0 |f(x)|}{\Gamma 1/s \, h(t_0)} \int_{t_0/\xi^{1/s}}^{\infty} \frac{h(\xi^{1/s} v)}{\xi^{1/s} v \, h(\xi^{1/s})} e^{-v^s} \xi^{1/s} dv.$$

$$\leq \frac{sH|f(x)|t_0}{\Gamma 1/s \, h(t_0)} \int_{t_0/\xi^{1/s}}^{\infty} \frac{h(v)}{v} \quad e^{-v^s} \quad dv$$

$$= o \, (1), \quad \xi \to 0+.$$

Adding the bounds, we have

$$|L(x; \xi ; s) - f(x) \, | = O \, (\frac{h(\xi^{1/s})}{\xi^{1/s}})$$

which completes the proof of theorem (5.2.1).

<u>Proof of theorem</u> (5.2.2) :

We have

$$\frac{\xi^{1/s}}{g(\xi^{1/s})} \int_{-\infty}^{\infty} |L(x; \xi ; s) - f(x) \, | \, dx$$

$$\leq \frac{s}{\Gamma 1/s \; g(\xi^{1/s})} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} |\phi_x(t)| \; e^{-|t|^s/\xi} \; dt.$$

$$\leq \frac{s}{\Gamma 1/s \; g(\xi^{1/s})} \int_{0}^{\infty} e^{-|t|^s/\xi} \; dt \int_{-\infty}^{\infty} |\phi_x(t)| \; dx.$$

$$= \frac{s}{\Gamma 1/s \; g(\xi^{1/s})} \int_{0}^{\infty} e^{-|t|^s/\xi} \lambda(t) \; dt.$$

We can write, as in the proof of theorem (5.2.1)

$$I_1 = O(1)$$

and also,

$$I_2 + I_3 + I_4 = \frac{s}{\Gamma 1/s \; g(\xi^{1/s})} \int_{t_0}^{\infty} \lambda(t) \; e^{-|t|^s/\xi} \; dt.$$

$$\leq \frac{s \, M \, t_0}{\Gamma 1/s \; g(t_0)} \int_{t_0}^{\infty} \frac{g(t)}{t \; g(\xi^{1/s})} \; e^{-|t|^s/\xi} \; dt.$$

$$= O(1), \quad \xi \to 0+.$$

Therefore,

$$I_1 + I_2 + I_3 + I_4 = O(1) + O(1)$$
$$= O(1).$$

Hence,

$$\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)| \; dx = O\left(\frac{g(\xi^{1/s})}{\xi^{1/s}}\right)$$

which completes the proof of theorem (5.2.2).

<u>5.3.</u> In this section, we have considered the conditions $O(\frac{h(t)}{t})$ and $O(\frac{g(t)}{t})$ on $\phi_x(t)$ and obtained the degree of approximation as before.

In this regard, we proved the following theorems:

*<u>Theorem</u> (5.3.1) : If $f(x) \in h(t)$, then

$$| L\,(x; \xi; o) - f(x) | = O\left(\frac{h(\xi^{1/s})}{\xi^{2/s}} \right)$$

provided $h(t)$ satisfies the following conditions:

$$\left(\frac{\xi^{1/s}}{t_o} \right) \frac{h(t_o)}{h(\xi^{1/s})} \; e^{-|t_o|^s/\xi} = o\,(1)$$

$$\frac{d}{dt}\left(\frac{h(t)}{t} \right) = O\left(\frac{h(t)}{t^2} \right), \text{ and}$$

$$\int_{t_o/\xi^{1/s}}^{\infty} \frac{h(v)}{v^2} \, e^{-v^s} \, dv < +\infty .$$

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* Khan, Arman and S. Umar [19].

*Theorem (5.3.2) : If $f(x) \in g(t)$, then

$$\int_{-\infty}^{\infty} | L(x;\xi;s) - f(x) | \, dx = O\left(\frac{g(\xi^{1/s})}{\xi^{2/s}} \right)$$

Under the conditions,

$$\frac{\xi^{1/s}}{t_o} \frac{g(t_o)}{g(\xi^{1/s})} e^{-|t_o|^s/\xi} = o(1)$$

$$\frac{d}{dt}\left(\frac{g(t)}{t} \right) = O\left(\frac{g(t)}{t^2} \right), \text{ and}$$

$$\int_{t_o/\xi^{1/s}}^{\infty} \frac{h(v)}{v^2} e^{-v^s} dv < +\infty$$

Proof of theorem (5.3.1) :

We can write,

$$L(x;\xi;s) - f(x) = \frac{s}{\Gamma 1/s \ \xi^{1/s}} \int_0^{\infty} \phi_x(t) \ e^{-|t|^s/\xi} \ dt.$$

on dividing both sides by $\dfrac{h(\xi^{1/s})}{\xi^{2/s}}$, we have

$$\frac{\xi^{2/s}}{h(\xi^{1/s})}\left[L(x;\xi;s)-f(x)\right] = \frac{s\ \xi^{1/s}}{\sqrt{1/s}\,h(\xi^{1/s})}\int_0^{t_0} e^{-|t|^s/\xi}\,d\Phi(t)+$$

$$\frac{s\ \xi^{1/s}}{2\sqrt{1/s}\ h(\xi^{1/s})}\Big[\int_{t_0}^{\infty} f(x+t)e^{-|t|^s/\xi}\,dt+\int_{t_0}^{\infty} f(x-t)\,e^{-|t|^s/\xi}\,dt -$$

$$\int_{t_0}^{\infty} 2f(x)\ e^{-|t|^s/\xi}\,dt\Big].$$

$= I_1 + I_2 + I_3 + I_4$, say

Now,

$$I_1 = \frac{s\ \xi^{1/s}}{\sqrt{1/s}\ h(\xi^{1/s})}\Big[e^{-|t_0|^s/\xi}\ \Phi(t_0) - \int_0^{t_0}\Phi(t)d_t(e^{-|t|^s/\xi})\Big]$$

$= I_1' + I_1''$, say

Evaluating I_1', we write

$$I_1' = \frac{s}{\sqrt{1/s}}\Big(\frac{\xi^{1/s}}{t_0}\Big)\frac{h(t_0)}{h(\xi^{1/s})}\quad e^{-(|t_0|/\xi^{1/s})^s}\ \frac{t_0\ \Phi(t_0)}{h(t_0)}$$

$= o\ (1).$

Also,

$$I_1'' = \frac{s\,\xi^{1/s}}{\sqrt{1/s}\;h(\xi^{1/s})}\; o\left[\int_0^{t_0} \frac{h(t)}{t}\, d_t\left(e^{-|t|^s/\xi}\right)\right]$$

$$= o\left[\left(\frac{\xi^{1/s}}{t_0}\right)\frac{h(t_0)}{h(\xi^{1/s})}\; e^{-(|t_0|/\xi^{1/s})^s}\right] +$$

$$o\left[\int_0^{t_0}\left(\frac{\xi^{2/s}}{t_0^2}\right)\frac{h(t)}{h(\xi^{1/s})}\; e^{-(|t|/\xi^{1/s})^s}\, d\left(\frac{|t|}{\xi^{1/s}}\right)\right]$$

$$= o\ (1) + O\ (1)$$

$$= O\ (1),\quad \xi \to 0+.$$

Therefore,

$$I_1 = I_1' + I_1''$$

$$= o\ (1) + O\ (1)$$

$$= O\ (1)$$

Now,

$$|\,I_2| = \leq \frac{s\,\xi^{1/s}}{2\sqrt{1/s}\;h(\xi^{1/s})}\; e^{-|t_0|^s/\xi}\int_{t_0}^{\infty} |f(x+t)|\, dt\,.$$

$$\leq \frac{M\, s\, t_0}{2\sqrt{1/s}\;h(t_0)}\left(\frac{\xi^{1/s}}{t_0}\right)\frac{h(t_0)}{h(\xi^{1/s})}\; e^{-(|t_0|/\xi^{1/s})^s}$$

$= o\ (1).$

Similarly,

$$I_3 = o\ (1)$$

Again,

$$|I_4| \leq \frac{s|f(x)|\ \xi^{1/s}}{\Gamma 1/s\ h(\xi^{1/s})} \int_{t_0}^{\infty} e^{-|t|^s/\xi}\ dt.$$

$$\leq \frac{s|f(x)\ t_0^2}{\Gamma 1/s\ h(t_0)} \int_{t_0}^{\infty} \left(\frac{\xi^{1/s}}{t^2}\right) \frac{h(t)}{h(\xi^{1/s})}\ e^{-|t|^s/\xi} dt.$$

$$\leq \frac{s|f(x)|\ t_0^2}{\Gamma 1/s\ h(t_0)} \int_{t_0/\xi^{1/s}}^{\infty} \frac{h(v)}{v^2}\ e^{-v^s}\ dv.$$

$$= o\ (1)$$

Adding the bounds for I_1, I_2, I_3 and I_4, we have

$$|\ L(x; \xi; s) - f(x)| = O\left(\frac{h(\xi^{1/s})}{\xi^{1/s}}\right)$$

which completes the proof of the theorem (5.3.1).

<u>Proof of theorem</u> (5.3.2) :

We have,

$$\frac{\xi^{2/s}}{g(\xi^{1/s})} \int_{-\infty}^{\infty} | L(x; \xi ; s) - f(x)| \, dx$$

$$\leq \frac{s \; \xi^{1/s}}{\Gamma 1/s \; g(\xi^{1/s})} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} |\phi_x(t)| \; e^{-|t|^s/\xi} \; dt.$$

$$= \frac{s \; \xi^{1/s}}{\Gamma 1/s \; g(\xi^{1/s})} \int_{0}^{\infty} e^{-|t|^s/\xi} \lambda(t) \, dt.$$

We can write, as in the proof of theorem (5.3.1)

$$I_1 = 0 \, (1)$$

and also,

$$I_2+I_3+I_4 = \frac{s \; \xi^{1/s}}{\Gamma 1/s \; g(\xi^{1/s})} \int_{t_0}^{\infty} \lambda(t) \; e^{-|t|^s/\xi} \; dt.$$

$$\leq \frac{s \, M \, t_0^2}{\Gamma 1/s \; g(t_0)} \int_{t_0}^{\infty} \left(\frac{\xi^{1/s}}{t^2}\right) \frac{g(t)}{g(\xi^{1/s})} \; e^{-|t|^s/\xi} \, dt.$$

$$= 0 \, (1)$$

Therefore,

$$I_1 + I_2 + I_3 + I_4 = 0\,(1) + 0(1)$$

$$= 0\,(1)$$

Hence,

$$\int_{-\infty}^{\infty} \mid L(x;\xi;s) - f(x) \mid dx = 0\left(\frac{g(\xi^{1/s})}{\xi^{2/s}}\right)$$

which completes the proof of theorem (5.3.2).

<u>5.4.</u> We have the following corollaries for the conditions $0\,(h(t))$ and $O(g(t))$ on $\phi_x(t)$ of our theorems (5.2.1) and (5.2.2).

<u>Corollary</u> (5.4.1) : If $\dfrac{h(t)}{t^2} = \dfrac{g(t)}{t^2} = t^{\alpha}$, $o < \alpha < 1$, then,

(a) $|L(x;\xi;s) - f(x)| = 0\,(\xi^{1+\alpha/s})$

(b) $\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)| \,dx = 0\,(\xi^{1+\alpha/s})$

<u>Corollary</u> (5.4.2) : If $\dfrac{h(t)}{t^2} = \dfrac{g(t)}{t^2} = t^{\alpha-1/p}$, we have

(a) $|L(x;\xi;s) - f(x)| = O(\xi^{1/s+\alpha/s-1/sp})$

(b) $\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)|\,dx = O(\xi^{1/s+\alpha/s-1/sp})$

where $f(x) \in L^p$ $(p > 1)$.

<u>Corollary</u> (5.4.3) : If $\dfrac{h(t)}{t^2} = \dfrac{g(t)}{t^2} = \psi(t)\, t^{-1/p}$,

where $\psi(t)$ is a positive increasing function and $f(x) \in L^p$ $(p \geq 1)$, then

(a) $|L(x;\xi;s) - f(x)| = O(\xi^{1/s-1/sp}\, \psi(\xi^{1/s}))$

(b) $\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)|\,dx = O(\xi^{1/s-1/sp}\, \psi(\xi^{1/s}))$

<u>5.5.</u> Considering the conditions $O(\dfrac{h(t)}{t})$ and $O(\dfrac{g(t)}{t})$ on $\phi_x(t)$, we have the following corollaries of our theorems (5.3.1) and (5.3.2).

<u>Corollary</u> (5.5.1) : If $\dfrac{h(t)}{t^2} = \dfrac{g(t)}{t^2} = t^{\alpha}$, $0 < \alpha < 1$, then

(a) $| L(x;\xi;s) - f(x) | = O(\xi^{\alpha/s})$.

(b) $\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)| \, dx = O(\xi^{\alpha/s})$.

__Corollary__ (5.5.2): If $\frac{h(t)}{t^2} = \frac{g(t)}{t^2} = t^{\alpha-1/p}$, where $f(x) \in L^p$ $(p > 1)$, then

(a) $|L(x;\xi;s) - f(x)| = O(\xi^{\alpha/s - 1/sp})$.

(b) $\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)| \, dx = O(\xi^{\alpha/s - 1/sp})$.

__Corollary__ (5.5.3): If $\frac{h(t)}{t^2} = \frac{g(t)}{t^2} = \psi(t)\, t^{-1/p}$,

Where $\psi(t)$ is a positive increasing function and $f(x) \in L^p$ $(p > 1)$, then

(a) $|L(x;\xi;s) - f(x)| = O(\xi^{-1/sp}\, \psi(\xi^{1/s}))$

(b) $\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)| \, dx = O(\xi^{1/sp}\, \psi(\xi^{1/s}))$.

CHAPTER VI

ON THE DEGREE OF APPROXIMATION TO A FUNCTION BELONGING TO THE CLASSES h(t) AND g(t).

6.1. In the previous chapters, we have discussed the problems of approximation using various operators and determined the degree of approximation for the classes of functions $k(t)$, $h(t)$ and $g(t)$.

The following some researchers have already used the various singular integrals in their field of studies.

Natanson, I.P. [33] discussed the accuracy of continuous functions by singular integrals.

He has taken $f(t)$ to be a periodic function of period 2π and continuous such that,

$$f_n(x) = \int_{-\pi}^{\pi} f(t)\, \bar{\Phi}_n\,(t-x)\, dt.$$

Where $\bar{\Phi}_n(t)$ is of period 2π, positive even and such that,

$\int_{-\pi}^{\pi} \bar{\phi}_n(t)\, dt = 1$ and $\delta_n = \int_0^{\pi} t\, \bar{\phi}_n(t)\, dt$, tends to zero as $n \to \infty$ (this condition is satisfied by the Kernels of Fejér, Jackson, De La Vallée-Poussin and Poisson-Cauchy).

He proved the following theorem:

Theorem (A) : If $f(x)$ has modulus of continuity $w(\delta)$, where f(x) is cont. and 2π periodic function then,

$$| f_n(x) - f(x) | \leq 3\, w(\delta n).$$

6.2. It is well known that if $0 < \alpha < 1$, the necessary and sufficient condition for the periodic function $f(x)$ to belong to the class Lip α is that,

$| f(x) - \sigma n(x) | = 0 \left(\frac{1}{n^{\alpha}} \right)$ uniformly where $\sigma n(x)$ is the Fejér sum of order n of the Fourier series of f. For $\alpha = 1$, the result is no longer true, but Alexits, G [2] detemined that $f \in$ Lip 1 iff,

$| \tilde{f} - \tilde{\sigma} n | = 0 \left(\frac{1}{n} \right)$, where $\tilde{f}$ is the conjugate of f and $\tilde{\sigma} n$ is its Fejér sum. His main result is a unification and generalisation of these results in the following theorem.

Theorem (B) : Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$, then $f \in \mathrm{Lip}\,(\alpha, p)$ iff,

$\| \tilde{f} - \tilde{\sigma}_n \|_p = O(\frac{1}{n^\alpha})$, where $\| \cdot \|_p$ denotes the norm in the metric L^p.

In the foregoing chapter, we have used the singular integrals (named Picard and ostrowski) only for determining the degree of approximation of functions belonging to the classes $h(t)$ and $g(t)$.

6.3. In this section, we have taken the conditions $O(h(t))$ and $O(g(t))$ on $\phi_x(t)$ and obtained the degree of approximation for Picard singular integral.

In fact, we have the following theorems:

***Theorem** (6.3.1): If $f(x) \in h(t)$, then

$$| f(x;\xi) - f(x) | = O\left(\frac{h(\xi)}{\xi}\right), \ \xi \to 0+.$$

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* Khan, Arman and S. Umar [20].

provided $h(t)$ satisfies the following conditions:

$$\frac{h(t_o)}{h(\xi)} \, e^{-|t_o|/\xi} = o(1)$$

$$\frac{d}{dt}(h(t)) = O\left(\frac{h(t)}{t}\right), \text{ and}$$

$$\int_{t_o/\xi}^{\infty} \frac{h(v)}{v} \, e^{-v} \, dv < +\infty$$

Where, we break the interval (o, ∞) into (o, t_o) and (t_o, ∞).

*<u>Theorem</u> (6.3.2) : If $f(x) \, @ \, g(t)$, then

$$\int_{-\infty}^{\infty} | F(x; \xi) - f(x) | \, dx = O\left(\frac{g(\xi)}{\xi}\right), \; \xi \to o+ ,$$

under the following conditions:

$$\frac{g(t_o)}{g(\xi)} \, e^{-|t_o|/\xi} = o(1),$$

$$\frac{d}{dt}(g(t)) = O\left(\frac{g(t)}{t}\right), \text{ and}$$

$$\int_{t_o/\xi}^{\infty} \frac{g(v)}{v} \, e^{-v} \, dv < +\infty$$

Proof of theorem (6.3.1):

We know that

$$F(x;\xi) - f(x) = \frac{1}{\xi} \int_0^\infty \phi_x(t)\ e^{-|t|/\xi}\ dt.$$

on dividing both sides by $\frac{h(\xi)}{\xi}$, we have

$$\frac{\xi}{h(\xi)}\left[F(x;\xi) - f(x)\right] = \frac{1}{h(\xi)} \int_0^{t_0} e^{-|t|/\xi}\ d\bar{\Phi}(t) +$$

$$\frac{1}{2\,h(\xi)}\Big[\int_{t_0}^\infty f(x+t)\ e^{-|t|/\xi}\ dt + \int_{t_0}^\infty f(x-t)\ e^{-|t|/\xi}\ dt -$$

$$\int_{t_0}^\infty 2\,f(x)\ e^{-|t|/\xi}\ dt\Big]$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say}$$

Evaluating I_1, we have

$$I_1 = \frac{1}{h(\xi)}\left[e^{-|t_0|/\xi}\ \bar{\Phi}(t_0) - \int_0^{t_0} \bar{\Phi}(t)\ d_t\left(e^{-|t|/\xi}\right)\right]$$

$$= I_{11} + I_{12}, \text{ say.}$$

Now,

$$I_{11} = \frac{h(t_o)}{h(\xi)} \quad e^{-|t_o|/\xi} \quad \frac{\bar{\phi}(t_o)}{h(t_o)}$$

$$= o\ (1).$$

Next,

$$I_{12} = \frac{1}{h\ (\xi)} \ O[\int_0^{t_o} h(t)\ d_t \left(e^{-|t|/\xi}\right)]$$

$$= O\ [\frac{h(t_o)}{h(\xi)} \quad e^{-|t_o|/\xi}\ ] + O[\int_0^{t_o} (\frac{\xi}{t}) \frac{h(t)}{h(\xi)}\ e^{-|t|/\xi}\ d(\frac{t}{\xi})]$$

$$= o\ (1) + O\ (1)$$

$$= O\ (1).$$

Therefore,

$$I_1 = I_{11} + I_{12}$$

$$= o\ (1) + O\ (1)$$

$$= O\ (1)$$

Again,

$$|\ I_2| \leq \frac{1}{2\ h(\xi)} \quad e^{-|t_o|/\xi} \int_{t_o}^{\infty} |\ f(x+t)\ |\ dt.$$

$$\leq \frac{M}{2\,h(t_0)} \quad \frac{h(t_0)}{h(\xi)} \quad e^{-|t_0|/\xi}$$

$$= O(1), \quad \xi \to 0+.$$

Similarly,

$$I_3 = o(1)$$

Also,

$$|I_4| \leq \frac{|f(x)|}{h(\xi)} \int_{t_0}^{\infty} e^{-|t|/\xi}\, dt.$$

$$\leq \frac{|f(x)|\, t_0}{h(t_0)} \int_{t_0}^{\infty} \frac{h(t)}{t h(\xi)} \, e^{-|t|/\xi}\, dt.$$

$$\leq \frac{|f(x)|\, t_0}{h(t_0)} \int_{t_0/\xi}^{\infty} \frac{h(v)}{v} \, e^{-v}\, dv$$

$$= O(1), \quad \xi \to 0+.$$

Hence,

$$|F(x;\xi) - f(x)| = O\left(\frac{h(\xi)}{\xi}\right)$$

which completes the proof of theorem (6.3.1).

<u>Proof of theorem</u> (6.3.2):

We have,

$$\frac{\xi}{g(\xi)} \int_{-\infty}^{\infty} |F(x;\xi) - f(x)|\,dx \leq \frac{1}{g(\xi)} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} |\phi_x(t)|\, e^{-|t|/\xi}\, dt.$$

$$= \frac{1}{g(\xi)} \int_{0}^{\infty} e^{-|t|/\xi}\, dt \int_{-\infty}^{\infty} |\phi_x(t)|\, dx$$

$$= \frac{1}{g(\xi)} \int_{0}^{\infty} e^{-|t|/\xi}\, \lambda(t)\, dt.$$

Therefore,

$I_1 = O(1)$, as in the proof of theorem (6.3.1).

and also,

$$I_2 + I_3 + I_4 = \frac{1}{2\, g(\xi)} \int_{t_0}^{\infty} \lambda(t)\; e^{-|t|/\xi}\; dt.$$

$$\leq \frac{H\, t_0}{2\, g(t_0)} \int_{t_0}^{\infty} \frac{g(t)}{t\, g(\xi)}\; e^{-|t|/\xi}\; dt.$$

$$= O(1).$$

Adding the bounds, we have

$$\int_{-\infty}^{\infty} |F(x;\xi) - f(x)|\, dx = O\left(\frac{g(\xi)}{\xi}\right),\ \xi \to 0+.$$

which completes the proof of theorem (6.3.2).

6.4. In this section, we used the operator (named ostrowski's operator) and determined the degree of approximation for the some classes $h(t)$ and $g(t)$, considering the conditions $o(h(t))$ and $O(g(t))$ on $\phi_x(t)$.

In this regard, we prove the following theorems:

**Theorem (6.4.1): If $f(x) \in h(t)$, then

$$| A(x;\xi) - f(x) | = O\left(\frac{h(\xi)}{\xi^2}\right)$$

provided $h(t)$ satisfies the following conditions.

$$\frac{h(t_0)}{h(\xi)} \quad e^{-(|t_0|/\xi)^{1/\xi}} = o(1), \text{ and}$$

$$\frac{d}{dt}(h(t)) = O\left(\frac{h(t)}{t}\right), \text{ and}$$

$$\int_{t_0/\xi}^{\infty} \frac{h(v)}{v} \quad e^{-v^{1/\xi}} dv < +\infty$$

**Theorem (6.4.2): If $f(x) \in g(t)$, then

**Khan, Arman and S. Umar [17].

$$\int_{-\infty}^{\infty} |A(x;\xi) - f(x)|\, dx = O\left(\frac{g(\xi)}{\xi^2}\right).$$

under the following conditions:

$$\frac{g(t_0)}{g(\xi)}\, e^{-(|t_0|/\xi)^{1/\xi}} = o(1),$$

$$\frac{d}{dt}(g(t)) = O\left(\frac{g(t)}{t}\right), \text{ and}$$

$$\int_{t_0/\xi}^{\infty} \frac{g(v)}{v}\, e^{-v^{1/\xi}}\, dv < +\infty$$

Proof of theorem (6.4.1):

We can write

$$A(x;\xi) - f(x) = \frac{1}{\sqrt{\xi}\,\xi^2} \int_0^{\infty} \phi_x(t)\, e^{-(|t|/\xi)^{1/\xi}}\, dt.$$

on dividing both sides by $\frac{h(\xi)}{\xi^2}$, we have

$$\frac{\xi^2}{h(\xi)}\left[A(x;\xi) - f(x)\right] = \frac{1}{\sqrt{\xi}\, h(\xi)} \int_0^{t_0} e^{-(|t|/\xi)^{1/\xi}}\, d\Phi(t) +$$

$$\frac{1}{2\sqrt{\xi}\, h(\xi)}\left[\int_{t_0}^{\infty} f(x+t)\, e^{-(|t|/\xi)^{1/\xi}}\, dt + \int_{t_0}^{\infty} f(x-t)\, e^{-(|t|/\xi)^{1/\xi}}\right.$$

$$- \int_{t_0}^{\infty} 2\, f(x)\, e^{-(|t|/\xi)^{1/\xi}}\, dt \,]$$

$= I_1 + I_2 + I_3 + I_4,$ say

For evaluating I_1, we write

$$I_1 = \frac{1}{\sqrt{\xi}\, h(\xi)} [\, e^{-(|t_0|/\xi)^{1/\xi}}\, \Phi(t_0) - \int_0^{t_0} \Phi(t)\, d_t (e^{-(|t|/\xi)^{1/\xi}})]$$

$= I_1' + I_1''$, say.

Again,

$$I_1' = \frac{1}{\sqrt{\xi}}\, \frac{h(t_0)}{h(\xi)}\, e^{-(|t|/\xi)^{1/\xi}}\, \frac{\Phi(t_0)}{h(t_0)}$$

$= o\,(1),\ \xi \to 0+.$

Also, we have

$$I_1'' = \frac{1}{\sqrt{\xi}\, h(\xi)}\, O\, [\int_0^{t_0} h(t)\, d_t\, (\, e^{-(|t|/\xi)^{1/\xi}})]$$

$$= O[\, \frac{h(t_0)}{h(\xi)}\, e^{-(|t_0|/\xi)^{1/\xi}}\,] + O[\int_0^{t_0} (\frac{\xi}{t})\, \frac{h(t)}{h(\xi)}$$

$$e^{-(|t|/\xi)^{1/\xi}}\, d(\frac{1}{\xi})]$$

$$= o\ (1) + O\ (1)$$

$$= O\ (1)$$

Adding the bounds, we get

$$I_1 = I_1' + I_1''$$

$$= o\ (1) + O\ (1)$$

$$= O\ (1),\ \xi \to 0+.$$

Again,

$$|I_2| \leq \frac{1}{2\sqrt{\xi}\, h(\xi)}\ e^{-(|t_0|/\xi)^{1/\xi}} \int_{t_0}^{\infty} |f(x+t)|\ dt.$$

$$\leq \frac{M}{2\sqrt{\xi}\ h(t_0)}\ \frac{h(t_0)}{h(\xi)}\ e^{-(|t_0|/\xi)^{1/\xi}}$$

$$= o\ (1),\quad \xi \to 0+.$$

Similarly,

$$|I_3| \leq \frac{1}{2\sqrt{\xi}\, h(\xi)}\ e^{-(|t_0|/\xi)^{1/\xi}} \int_{t_0}^{\infty} |\ f(x-t)\ |\ dt.$$

$$= o\ (1)$$

Next,

$$|I_4| \leq \frac{1}{\sqrt{\xi}\ h(\xi)} \int_{t_0}^{\infty} e^{-(|t|/\xi)^{1/\xi}}\ dt.$$

$$\leq \frac{|f(x)|t_0}{\sqrt{\xi}\, h(t_0)} \int_{t_0}^{\infty} \frac{h(t)}{t\, h(\xi)} e^{-(|t|/\xi)^{1/\xi}} dt.$$

$$= \frac{|f(x)|t_0}{\sqrt{\xi}\, h(t_0)} \int_{t_0/\xi}^{\infty} \frac{h(v)}{v} e^{-v^{1/\xi}} dv.$$

$$= o(1).$$

Adding the bounds, we have

$$I_1 + I_2 + I_3 + I_4 = O(1)$$

Hence

$$|A(x;\xi) - f(x)| = O\left(\frac{h(\xi)}{\xi^2}\right), \quad \xi \to 0+.$$

which completes the proof of the theorem (6.4.1).

<u>Proof of theorem</u> (6.4.2):

We have

$$\frac{\xi^2}{g(\xi)} \int_{-\infty}^{\infty} |A(x;\xi) - f(x)|\, dx$$

$$\leq \frac{1}{\sqrt{\xi}\, g(\xi)} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} |\phi_x(t)| e^{-(|t|/\xi)^{1/\xi}} dt.$$

$$= \frac{1}{\sqrt{\xi}\, g(\xi)} \int_{0}^{\infty} e^{-(|t|/\xi)^{1/\xi}} dt \int_{-\infty}^{\infty} |\phi_x(t)|\, dx$$

$$= \frac{1}{\sqrt{\xi}\, g(\xi)} \int_{0}^{\infty} e^{-(|t|/\xi)^{1/\xi}} \lambda(t)\, dt.$$

We can write, as in the proof of theorem (6.4.1).

$$I_1 = O(1)$$

and also,

$$I_2 + I_3 + I_4 = \frac{1}{\sqrt{\xi}\, g(\xi)} \int_{t_0}^{\infty} \lambda(t)\, e^{-(|t|/\xi)^{1/\xi}}\, dt.$$

$$\leq \frac{K\, t_0}{\sqrt{\xi}\, g(t_0)} \int_{t_0}^{\infty} \frac{g(t)}{t\, g(\xi)}\, e^{-(|t|/\xi)^{1/\xi}}\, dt.$$

$$= O(1)$$

Adding the bounds, we have

$$I_1 + I_2 + I_3 + I_4 = O(1)$$

Hence,

$$\int_{-\infty}^{\infty} |A(x;\xi) - f(x)|\, dx = O\left(\frac{g(\xi)}{\xi^2}\right), \quad \xi \to 0+$$

which completes the proof of theorem (6.4.2).

__6.5.__ We have the following corollaries for our theorems (6.3.1) and (6.3.2) on considering the conditions

$O\left(\frac{h(t)}{t}\right)$ and $O\left(\frac{g(t)}{t}\right)$ on $\phi_x(t)$ in place of $O(h(t))$ and $O(g(t))$ respectively.

Corollary (6.5.1): If $f(x) \in h(t)$, then

$$|F(x;\xi) - f(x)| = O\left(\frac{h(\xi)}{\xi^2}\right)$$

provided $h(t)$ satisfies the following conditions

$$\left(\frac{\xi}{t_0}\right)\frac{h(t_0)}{h(\xi)} e^{-(|t_0|/\xi)} = o(1),$$

$$\frac{d}{dt}\left(\frac{h(t)}{t}\right) = O\left(\frac{h(t)}{t^2}\right), \text{ and}$$

$$\int_{t_0/\xi}^{\infty} \frac{h(v)}{v^2} e^{-v}\, dv < +\infty$$

Corollary (6.5.2): If $f(x) \in g(t)$, then

$$\int_{-\infty}^{\infty} |F(x;\xi) - f(x)|\, dx = O\left(\frac{g(\xi)}{\xi^2}\right)$$

Under the following conditions;

$$\left(\frac{\xi}{t_0}\right) \frac{g(t_0)}{g(\xi)} \quad e^{-(|t_0|/\xi)} \quad = o\ (1),$$

$$\frac{d}{dt}\ \left(\frac{g(t)}{t}\right) = O\ \left(\frac{g(t)}{t^2}\right), \text{ and}$$

$$\int_{t_0/\xi}^{\infty} \frac{g(v)}{v^2} \quad e^{-v} \quad dv < +\infty.$$

Where, we break the interval (o, ∞) into $(0, t_o)$ and (t_o, ∞).

The proofs are similar to those of theorems (6.3.1) and (6.3.2).

__6.6.__ Special cases of theorems (6.3.1) and (6.3.2)

(1). If $\dfrac{h(t)}{t^2} = \dfrac{g(t)}{t^2} = t^{\alpha}$, $o < \alpha < 1$, then

(a) $|F(x; \xi) - f(x)| = O\ (\xi^{1+\alpha})$

(b) $\int_{-\infty}^{\infty} |F(x; \xi) - f(x)|\ dx = O\ (\xi^{1+\alpha})$

(2) If $\dfrac{h(t)}{t^2} = \dfrac{g(t)}{t^2} = t^{\alpha-1/p}$, where $f(x) \in L^p (p > 1)$,

we have

(a) $|P(x; \xi) - f(x)| = O(\xi^{\alpha+1-1/p})$

(b) $\int_{-\infty}^{\infty} |P(x; \xi) - f(x)|\, dx = O(\xi^{\alpha+1-1/p})$

(3) If $\frac{h(t)}{t^2} = \frac{g(t)}{t^2} = \psi(t)\, t^{-1/p}$, where $\psi(t)$ is a positive increasing function and $f(x) \in L^p$ $(p > 1)$, then

(a) $|P(x; \xi) - f(x)| = O(\xi^{1-1/p}\, \psi(\xi))$.

(b) $\int_{-\infty}^{\infty} |P(x; \xi) - f(x)|\, dx = O(\xi^{1-1/p}\, \psi(\xi))$

6.7. We have the following corollaries for ostrowski's operator

Corollary (6.7.1): If $\frac{h(t)}{t^2} = \frac{g(t)}{t^2} = t^{\alpha}$, $0 < \alpha < 1$, then

(a) $|A(x; \xi) - f(x)| = O(\xi^{\alpha})$

(b) $\int_{-\infty}^{\infty} |A(x; \xi) - f(x)|\, dx = O(\xi^{\alpha})$

Corollary (6.7.2) : If $\frac{h(t)}{t^2} = \frac{g(t)}{t^2} = t^{\alpha-1/p}$, $(p > 1)$

then

(a) $|A(x; \xi) - f(x)| = O(\xi^{\alpha-1/p})$

(b) $\int_{-\infty}^{\infty} |A(x; \xi) - f(x)| \, dx = O(\xi^{\alpha-1/p})$

Corollary (6.7.3): If $\frac{h(t)}{t^2} = \frac{g(t)}{t^2} = \psi(t)\, t^{-1/p}$

Where $\psi(t)$ is positive increasing function and $f(x) \in L^p$ $(p > 1)$, then

(a) $|A(x; \xi) - f(x)| = O(\xi^{-1/p} \psi(\xi))$

(b) $\int_{-\infty}^{\infty} |A(x; \xi) - f(x)| \, dx = O(\xi^{-1/p} \psi(\xi))$.

A P P E N D I X - A

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A P P E N D I X – B

With the Compliment of the Author

A NOTE ON THE ORDER OF APPROXIMATION OF GAUSS-WEIERSTRASS SINGULAR INTEGRAL

BY

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Reprinted from

SOOCHOW JOURNAL OF MATHEMATICS

Volume 6, pp. 145-147, December 1980

Science College, Soochow University

TAIPEI, TAIWAN, REPUBLIC OF CHINA

SOOCHOW JOURNAL OF MATHEMATICS
Volume 6, December 1980

A NOTE ON THE ORDER OF APPROXIMATION OF GAUSS-WEIERSTRASS SINGULAR INTEGRAL

BY

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1. **Introduction.** Let $f(x)$ be a periodic integral function $(L^p,\ p \geq 1)$ with period 2π and let

$$(1.1) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(x)$$

be its Fourier series.

The Gauss-Weierstrass singular integral of $f(x)$ is defined by

$$W_n(x) = W(x;\ \xi) = \sum_{k=0}^{\infty} e^{-(k^2\xi/4)} A_k(x)$$
$$= \sqrt{\pi/\xi} \int_{-\pi}^{\pi} f(x+t)\, e^{(-t^2/\xi)}\, dt + o(\xi),$$

where $o(\xi) \to 0$ as $\xi \to 0$.

Definition (1.1). A function $f(x)$, integrable L, is said to belong to $k(t)$ class, where $k(t)$ is positive increasing function and $k(t)/t$ is decreasing such that

(a) $|f(x+t) - f(x)| = O(k(t))$

(b) $k(xy) = k(x)\, k(y)$.

We notice that by taking

(i) $k(t) = t^{\alpha}$, $0 < \alpha < 1$, then our class reduces to Lip α.

(ii) $k(t) = t^{\alpha - 1/p}$ and if $f(x) \in L^p (p > 1)$, then our class reduces to Lip$(\alpha,\ p)$ (Hardy and Littlewood [1]).

(iii) $k(t) = \psi(t)\ t^{-1/p}$, $\psi(t)$ is positive increasing function and $f(x) \in L^p(p > 1)$, then our class reduces to $(\psi(t),\ p)$ class (Huzoor H. Khan [2]).

We define

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$$\phi(t) = f(x+t) + f(x-t) - 2f(x) .$$

2. Our main purpose here is to determine the order of approximation by considering Gauss-Weierstrass singular integrals of the Fourier series of $f(x)$ for a more general class $k(t)$.

Our theorem is as follows:

Theorem (2.1). If $f(x)$ belongs to $k(t)$, then

$$|W(x;\xi) - f(x)| = O(\xi^{1/2} k(\xi)) ,$$

provided $k(t)$ satisfies the following

$$\int_{\sqrt{\xi}}^{\pi/\sqrt{\xi}} \frac{k(t)}{t^2} dt = O\left(\frac{k(\xi^{1/2})}{\xi}\right), \qquad \xi \longrightarrow 0 .$$

Proof. Since

$$\begin{aligned} W(x;\ \xi) - f(x) &= \sqrt{\pi/\xi} \int_{-\pi}^{\pi} |f(x+t) + f(x-t) - 2f(x)| e^{(-t^2/\xi)} dt \\ &\quad + R(x,\ \xi) \\ &= 2\sqrt{\pi/\xi} \int_0^{\pi} \phi(t)\, e^{(-t^2/\xi)} dt + o(\xi) \\ &\quad \text{(following Sunouchi and Watari [3])} \\ &= 2\sqrt{\pi/\xi} \left[\int_0^{\xi} + \int_{\xi}^{\pi}\right] \phi(t)\, e^{(-t^2/\xi)} dt \\ &= I_1 + I_2, \qquad \text{say.} \end{aligned}$$

Now evaluating I_1, we have

$$I_1 = 2\sqrt{\pi/\xi} \int_0^{\xi} \phi(t)\, e^{(-t^2/\xi)} dt$$

$$\begin{aligned} |I_1| &\leq 2\sqrt{\pi/\xi} \int_0^{\xi} |\phi(t)|\, e^{(-t^2/\xi)} dt \\ &= 2\sqrt{\pi/\xi} \int_0^{\xi} k(t)\, e^{(-t^2/\xi)} dt \\ &\leq 2\sqrt{\pi/\xi}\, k(\xi) \int_0^{\xi} e^{(-t^2/\xi)} dt \end{aligned}$$

Now putting $t/\sqrt{\xi} = u$, we have

$$= O(k(\xi)) \int_0^{\sqrt{\xi}} e^{-u^2} du$$

$$I_1 = O(k(\xi))\ O(\xi^{1/2}) = O(\xi^{1/2} k(\xi)) .$$

Next,

$$I_2 = 2\sqrt{\pi/\xi} \int_{\xi}^{\pi} \phi(t)\, e^{(-t^2/\xi)} dt$$

$$|I_2| \leq 2\sqrt{\pi/\xi} \int_{\xi}^{\pi} |\phi(t)|\, e^{(-t^2/\xi)}\, dt$$

$$= 2\sqrt{\pi/\xi} \int_{\xi}^{\pi} k(t)\, e^{(-t^2/\xi)}\, dt\,.$$

Now putting $t/\sqrt{\xi} = u$, we have

$$= O(k(\xi)) \int_{\sqrt{\xi}}^{\pi/\sqrt{\xi}} e^{-u^2} k(u)\, du$$

$$= O(k(\xi)) \int_{\sqrt{\xi}}^{\pi/\sqrt{\xi}} u^{-2} u^2 e^{-u^2} k(u)\, du$$

$$= O(k(\xi))\, O(\xi) \int_{\sqrt{\xi}}^{\pi/\sqrt{\xi}} u^{-2} k(u)\, du$$

$$= O(k(\xi))\, O(\xi)\, O\left(\frac{k(\xi^{1/2})}{\xi}\right)$$

$$I_2 = O(\xi^{1/2} k(\xi))\,.$$

Adding the bounds for I_1 and I_2, we have

$$|W(x;\ \xi) - f(x)| = O(\xi^{1/2} k(\xi))\,,$$

which terminates the proof.

Remark. It may also be remarked that by giving different values to $k(t)$, we get some interesting results.

(i) If $k(t) = t^{\alpha}$, we have

$$|W(x;\ \xi) - f(x)| = O(\xi^{\alpha+1/2})$$

(ii) If $k(t) = t^{\alpha-1/p}$, we get

$$|W(x;\ \xi) - f(x)| = O(\xi^{\alpha+1/2-1/p})$$

(iii) If $k(t) = \psi(t)\ t^{-1/p}$, we obtain

$$|W(x;\ \xi) - f(x)| = O(\xi^{1/2-1/p}\, \psi(\xi))\,.$$

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